On the equivalence between traction- and stress-based approaches for the modeling of localized failure in solids

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Abstract

This work investigates systematically traction- and stress-based approaches for the modeling of strong/regularized discontinuities induced localized failure in solids. Two complementary methodologies, i.e., *traction-based discontinuities localized in an elastic solid* and *strain localization of a stress-based inelastic softening solid*, are addressed. In the former it is assumed *a priori* that the discontinuity (band) forms with a continuous stress field and along the known orientation. A traction-based failure criterion is introduced to characterize the discontinuity (band) and the orientation is determined from Mohr's maximization postulate. If the (apparent) displacement jumps are retained as independent variables, the strong/regularized discontinuity approaches follow, requiring constitutive models for both the bulk and discontinuity (band). Elimination of the displacement jumps at the material point level results in the embedded/smeared discontinuity approaches in which an overall inelastic constitutive model fulfilling the static constraint suffices. The second methodology is then adopted to check whether the assumed strain localization can occur and identify its consequences on the resulting approaches. The kinematic constraint guaranteeing stress boundedness/continuity upon strain localization is established for general inelastic softening solids. Application to a unified elastoplastic damage model naturally yields all the ingredients of a localized model for the discontinuity (band), justifying the first methodology. Two dual but not necessarily equivalent approaches, i.e., the traction-based elastoplastic damage model and the stress-based projected discontinuity model, are identified. The former is equivalent to the embedded/smeared discontinuity approaches, whereas in the later the discontinuity orientation and associated failure criterion, not given *a priori*, are determined consistently from the kinematic constraint. The *bi-directional* connections and equivalence conditions between the traction- and stress-based approaches are classified. Closed-form results under plane stress condition are also given. A generic failure criterion of either elliptic, parabolic or hyperbolic type, is analyzed in a unified manner, with the classical von Mises (J_2), Drucker-Prager, Mohr-Coulomb and many other frequently employed criteria recovered as its particular cases.

Keywords:
Localized failure; strain localization; constitutive behavior; discontinuities; fracture; plasticity; damage.

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1. Introduction

Overall responses of inelastic softening solids are characterized by strain localization, i.e. a manifestation of concentration of micro-structural defects. Depending on the material interested, the phenomena resulting from strain localization may be diverse: dislocations of order of microns in crystal metals, cracks of order of millimeters in concrete, and shear bands of order ranging from millimeters to kilometers in granular and geological problems. From the structural point of view these localization band may be regarded as a fracture surface of small or even negligible width compared to the length scale of the structure. Structural collapse is often induced by formation of such localization bands. Therefore, it is of utmost significance to evaluate (residual) structural safety once strain localization occurs, and to prevent potential catastrophic collapse caused by localized failure. However, despite the recent progresses made, the modeling of strain localization and subsequent structural collapse still remains a challenging issue.

Strain localization inevitably induces strain/displacement discontinuities, hindering the applicability of classical continuum mechanics. With respect to the strategies for the approximation of such discontinuities and resulting consequences on material or structural responses, different approaches have been proposed ever since the pioneering work of Ngo and Scordelis [34] and Rashid [48]. Generally speaking, existing approaches can be classified into the nonlinear fracture mechanics based discontinuous models or the generalized continuum mechanics based continuous models. In the computational context, they correspond to the discrete and smeared discontinuity methods, respectively.

In the discontinuous (discrete) approach strain/displacement jumps are explicitly accounted for by embedding the discontinuities into a solid matrix along preferred orientations. It is generally assumed that energy dissipation mechanisms are localized into the discontinuities while the bulk remains elastic. The traction continuity condition is imposed between them. The overall inelastic behavior is of anisotropy by construction. To characterize the dissipative behavior lumped in the discontinuities, vectorial traction-based cohesive models furnished with the fracture energy are introduced. Generally, displacement discontinuities are regarded as zero-width failure surfaces characterized by tractions vs. displacement jumps [3, 4, 17, 22]. Alternatively, strain discontinuities across the localized band with a finite width can be represented in terms of tractions vs. inelastic deformations (i.e., apparent displacement jumps normalized with respect to the bandwidth) [9, 10]. Depending on the recoverable/irreversible properties of the discontinuities, traction-based cohesive models of either plastic [7, 59, 62], damage [1, 2, 24] or combined plastic-damage [63, 67] type can be established.

Contrariwise, the continuous approach relies on the introduction of a generalized continuum, so that stress-based constitutive models with regularized softening regime can be used. In this approach, the strain/displacement discontinuities are regularized (smeared) so that the classical concepts of (average) stress and strain still apply. It is no longer necessary to make distinction between the elastic bulk and the inelastic localization band. But rather, the overall nonlinear behavior of the weakened medium is described by generalized constitutive laws in terms of stress vs. strain tensors equipped with softening internal variables. In this way, plasticity and damage mechanics or their combination [15, 25] can be employed to develop appropriate inelastic constitutive laws. Induced anisotropy can be considered either theoretically [19, 30, 67] or in the computational context [12, 13]. Furthermore, to guarantee objectivity of the energy dissipated during the fracture process, the softening regime has to be regularized with respect to the length scale of localization band. The fracture energy – a material property measuring the dissipation per fracture surface, and an appropriately identified localization band width [5, 36], are fundamental ingredients for this purpose.

In the traction-based approached for the modeling of localized failure in solids, a crucial step is to determine the discontinuity orientation. This is a non-trivial goal for a new or propagating discontinuity whose orientation is not pre-defined or known *a priori*. To this end, *ad hoc* strategies have to be introduced, usually in an heuristic manner. For instance, the maximum tensile stress (i.e., Rankine) criterion is often adopted for mode I failure in quasi-brittle materials. For more general cases, selecting the discontinuity orientation according to some makeshift condition and fixing it afterwards becomes a superimposed condition on the material behavior. Fortunately, the occurrence of necessary requisites for a certain type of failure to be initiated during the whole deformation process in inelastic solids provides useful information. In this aspect, the pioneering works by Hill [20, 21], Thomas [61] and Rice [49, 54] have been widely adopted in the literature. The necessary conditions for discontinuous bifurcation and strain localization of elastoplastic materials were identified and formulations for the orientation of shear bands were obtained; see [55] and the references therein. Discontinuous bifurcation analysis, based on the assumption of linear comparison solid (inelastic loading state both inside and outside the localization band) and the traction continuity condition, has now become the standard tool to analyze propagation of strain or weak discontinuities.

For strong discontinuities in solids with strain softening regimes, there is no consensus for the determination of the discontinuity orientation. For instance, Simó et al. [58] and Oliver [37] suggested using the discontinuous bifurcation condition with perfect (null) softening/hardening modulus to determine the discontinuity orientation. This necessary discontinuous bifurcation condition in general does not guarantee the occurrence of strong discontinuities [38]. To remedy this problem, a variable band width model was proposed in the last reference, so that weak discontinuities can evolve smoothly to a strong one at a later stage. However, for strain localization to occur in a softening solid and develop eventually into a fully softened discontinuity at the final stage of the deformation process, material points inside the discontinuity (band) undergo inelastic loading while those outside it unload elastically. That is, the actual loading state is not consistent with the assumption of linear comparison solids. Therefore, the limit decohesion cannot be guaranteed in general cases and stress locking occurs due to the mis-prediction of the discontinuity orientation. Furthermore, due to the singular strain field associated with displacement discontinuities, traction continuity alone is not sufficient to guarantee physically meaningful results, but rather, stress boundedness also has to be invoked [37–39, 58]. Noticing this fact, Cervera et al. [11] recently used the kinematic compatibility condition resulting from stress boundedness to determine the discontinuity orientation, so that the stress locking-free property can be guaranteed for a fully softening discontinuity. The analytical results, obtained for von Mises (J_2) plastic materials in the cases of plane stress and plane strain, were validated by numerical simulations.

Failure criterion is another indispensable ingredient for the modeling of localized failure in solids. It defines an elastic domain outside which nonlinear energy dissipation mechanisms are active. In discontinuous approaches a traction-based failure criterion is in general adopted. For mode-I failure a trivial Rankine criterion can be employed. For mixed mode failure, the simple linear Mohr-Coulomb criterion is in general insufficient and more complex function has to be introduced, usually in an *ad hoc* manner. For instance, either elliptical [6, 24, 47, 67], parabolic [32, 35] or hyperbolic [7, 14, 33, 62] function has been postulated in the literature to describe mixed-mode failure in quasi-brittle materials; see Part II for more details. However, the involved parameters are essentially mesoscopical entities hard to be determined from rather limited experimental tests. This shortcoming restrains heavily the application of traction-based discontinuous approaches. On the other hand, situations are totally different for the continuous approaches. Large quantities of stress-based failure criteria have been suggested based on triaxial test data, especially for concrete-like quasi-brittler materials [15].

Traction-based discontinuity methods and Stress-based material models with regularized softening regime both have been widely adopted for the modeling of localized failure in solids, though they are motivated from different methodologies and usually regarded as unconnected ones. It would be of great significance to establish a solid theoretical connection between results obtained from strain localization analysis of stress-based inelastic materials and from traction-based discontinuities localized into the elastic matrix. Such a correspondence will serve as a guidance how to select an appropriate approach when modeling localized failure in solids. Furthermore, it will also bridge the gap in identifying those mesoscopic parameters involved in the traction-based failure criterion from the easily obtained macroscopic properties of materials. In the literature, some attempts have been made to link them each other. For instance, Oliver and coworkers [38–41] derived the traction-based discontinuity models by projecting inelastic stress-based constitutive laws onto the discontinuity orientation. They managed deriving in closed-form the traction-based failure criterion for the isotropic damage model in total form [39–41]. Except for the Rankine and plane strain von Mises criteria [38], stress-based plastic and plastic-damage models with general failure criterion have not been sufficiently accounted for. More importantly, the discontinuity orientation is determined through the discontinuous bifurcation condition. Accordingly, the resulting traction-based discontinuity model is in general inconsistent with the original stress-based counterpart and some mismatches are observed in the discontinuity kinematics [41, 42].

Recent investigations [64, 65] show that upon strain localization the stress field not only is bounded, but also is continuous, providing only the discontinuity modes caused by relative rigid body motions (translations and rotations) are considered [63]. Satisfaction of such a stress boundedness/continuity condition upon strain localization not only allows the formation of a fully softened discontinuity, but also establishes a bridge connecting the stress- and traction-based approaches motivated from rather different viewpoints. This work is to make further contributions to the above topics. Two complementary methodologies, i.e., *traction-based discontinuities localized in an elastic solid* and *strain localization of a stress-based inelastic solid*, are presented. Both strong and regularized discontinuities are considered. Regarding the strategies dealing with the orientation and failure criterion of the discontinuity, the connections and in particularly, the equivalence conditions, between traction- and stress-based approaches are investigated systematically.

Note that the theoretical connections are *bi-directional* rather than uni-directional as in the existing literature. For the sake of simplicity only localized models with associated evolution laws are considered in this work, though the basic principles also apply for non-associated cases.

This paper is organized as follows. After this introduction, the kinematics and governing equations of localized failure in solids are briefly presented in Section 2. Within the first methodology of discontinuities localized in an elastic solid, several traction-based approaches are discussed in Section 3. To justify the first methodology, an alternative one, i.e., strain localization of a stress-based inelastic solid, is then considered in Section 4. The kinematic constraint guaranteeing the stress boundedness/continuity condition is derived for general inelastic softening solids and applied to a unified elastoplastic damage model. Two dual (formally identical) but not necessarily equivalent approaches, i.e., the traction-based elastoplastic damage model and the stress-based projected discontinuity model, are identified. Section 5 addresses the *bi-directional* connections, in particular, the equivalence conditions, between the aforementioned traction- and stress-based approaches. In Section 6 closed-form results in plane stress condition are given, with respect to a generic stress-based failure criterion. Some relevant conclusions drawn in Section 7 close the paper.

Notation. Compact tensor notation is used in this paper as far as possible. As a general rule, scalars are denoted by italic light-face Greek or Latin letters (e.g. a or λ); vectors and second-order tensors are signified by italic boldface minuscule and majuscule letters like $\boldsymbol{a}$ and $\boldsymbol{A}$, respectively. Fourth-order tensors are identified by blackboard-bold majuscule characters (e.g. $\mathbb{A}$). Symbols $\boldsymbol{I}$ and $\mathbb{I}$ represent the second-order and symmetric fourth-order identity tensors, respectively. Superscripts '$^{\text{T}}$' and '$^{\text{sym}}$' indicate the transposition and symmetrization operations, respectively. The inner products with single and double contractions are denoted by '$\cdot$' and '$:$', respectively. The dyadic product '$\otimes$' and the symmetrized Kronecker product $\overline{\underline{\otimes}}$ are defined as

$$\big(\boldsymbol{A} \otimes \boldsymbol{B}\big)_{ijkl} = A_{ij} B_{kl}, \qquad \big(\boldsymbol{A} \,\overline{\underline{\otimes}}\, \boldsymbol{B}\big)_{ijkl} = \frac{1}{2}\big(A_{ik} B_{jl} + A_{il} B_{jk}\big)$$

2. General formulation of localized failure in solids

Let us consider the domain $\Omega \subset \mathbb{R}^{n_{\text{dim}}}$ ($n_{\text{dim}} = 1, 2, 3$) shown in Fig. 1. It is occupied by a solid with reference position vector $\boldsymbol{x} \in \mathbb{R}^{n_{\text{dim}}}$. The boundary is denoted by $\Gamma \subset \mathbb{R}^{n_{\text{dim}}-1}$, with an external unit normal vector $\boldsymbol{n}^*$. Deformations of the solid are characterized by the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^{n_{\text{dim}}}$ and the infinitesimal strain field $\boldsymbol{\epsilon} := \nabla^{\text{sym}} \boldsymbol{u}$, with $\nabla(\cdot)$ being the spatial gradient operator. The solid is subjected to a distributed body force $\boldsymbol{b}^* : \Omega \to \mathbb{R}^{n_{\text{dim}}}$ per unit volume. Surface tractions $\boldsymbol{t}^* : \Gamma_t \to \mathbb{R}^{n_{\text{dim}}}$ and displacements $\boldsymbol{u}^* : \Gamma_u \to \mathbb{R}^{n_{\text{dim}}}$ are imposed on the disjoint and complementary parts $\Gamma_t \subset \Gamma$ and $\Gamma_u \subset \Gamma$ of the boundary Γ, respectively.

At the early stage of the deformation process, standard compatibility relations of a continuum medium apply. That is, both the displacement and strain fields are continuous and regular (bounded). Upon satisfaction of a specific criterion, strain localization occurs, inevitably inducing strain/displacement jumps. To approximate these jumps, a strong discontinuity or a regularized one may be introduced. In either case, the standard kinematics no longer applies and is replaced by Maxwell's compatibility condition.

2.1. Kinematics of discontinuities

Displacement jumps can be described by a strong discontinuity. As depicted in Fig. 2(a), the strong discontinuity $\mathcal{S}$ splits the solid Ω into two parts Ω^+ and Ω^-, located "ahead of" and "behind" $\mathcal{S}$, respectively, in such a way that $\Omega^+ \cup \Omega^- \cup \mathcal{S} = \Omega$. The discontinuity orientation is denoted by a unit normal vector $\boldsymbol{n}$, pointing from Ω^- to Ω^+ and fixed along time (i.e., $\dot{\boldsymbol{n}} = \boldsymbol{0}$). The strong discontinuity $\mathcal{S}$ causes displacement jumps $\boldsymbol{w} := \boldsymbol{u}(\boldsymbol{x} \in \Omega^+ \cap \mathcal{S}) - \boldsymbol{u}(\boldsymbol{x} \in \Omega^- \cap \mathcal{S})$ across it. In this case, the displacement field $\boldsymbol{u}(\boldsymbol{x})$ can be expressed as

$$\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \mathscr{H}_{\mathcal{S}}(\boldsymbol{x})\, \hat{\boldsymbol{u}}(\boldsymbol{x}) \tag{2.1a}$$

so that the strain field $\boldsymbol{\epsilon}(\boldsymbol{x})$ is given by

$$\boldsymbol{\epsilon}(\boldsymbol{x}) := \nabla^{\mathrm{sym}}\boldsymbol{u}(\boldsymbol{x}) = \nabla^{\mathrm{sym}}\bar{\boldsymbol{u}}(\boldsymbol{x}) + \big(\boldsymbol{w} \otimes \boldsymbol{n}\big)^{\mathrm{sym}}\, \delta_{\mathcal{S}}(\boldsymbol{x}) \tag{2.1b}$$

where $\bar{\boldsymbol{u}}(\boldsymbol{x})$ denotes the continuous part of the displacement field; $\mathscr{H}_{\mathcal{S}}(\boldsymbol{x})$ is Heaviside function defined at the interface $\mathcal{S}$, i.e., $\mathscr{H}_{\mathcal{S}}(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \Omega^- \cup \mathcal{S}$ and $\mathscr{H}_{\mathcal{S}}(\boldsymbol{x}) = 1$ otherwise; $\delta_{\mathcal{S}}(\boldsymbol{x})$ denotes Dirac-delta at the discontinuity $\mathcal{S}$. By construction, the relative displacement field $\hat{\boldsymbol{u}}(\boldsymbol{x}) : \Omega \to \mathbb{R}^{n_{\mathrm{dim}}}$, satisfying the property $\hat{\boldsymbol{u}}(\boldsymbol{x} \in \mathcal{S}) = \boldsymbol{w}$, describes how the part Ω^+ moves as a relative rigid body (e.g. translations and rotations) with respect to the other one Ω^-; see Wu [63]. Accordingly, though the relative displacement field $\hat{\boldsymbol{u}}(\boldsymbol{x})$ itself is not necessarily constant, its contribution to the strain field vanishes, i.e., $\nabla^{\mathrm{sym}}\hat{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0}$. If only relative rigid body translations are considered, the relative displacement field $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is constant and coincident with the displacement jump $\boldsymbol{w}$.

The unbounded strain field (2.1b) resulting from the discontinuous displacement field (2.1a) can be regularized over a discontinuity band $\mathcal{B}$ of finite width b. Here, the width b is not a physical length but a regularization parameter which can be made as small as desired. As shown in Fig. 2(b), the regularized discontinuity (or discontinuity band) $\mathcal{B}$ is delimited by two surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$ parallel to the discontinuity $\mathcal{S}$, i.e., $\Omega^+ \cup \Omega^- \cup \mathcal{B} = \Omega$. In this case, the displacement field $\boldsymbol{u}(\boldsymbol{x})$ is continuous, with an apparent displacement jump $\boldsymbol{w} := \boldsymbol{u}(\boldsymbol{x} \in \Omega^+ \cap \mathcal{S}^+) - \boldsymbol{u}(\boldsymbol{x} \in \Omega^- \cap \mathcal{S}^-)$ across the discontinuity band $\mathcal{B}$. Accordingly, the $\mathcal{C}^0$-continuous displacement field $\boldsymbol{u}(\boldsymbol{x})$ can be expressed as

$$\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \mathscr{H}_{\mathcal{B}}(\boldsymbol{x})\, \hat{\boldsymbol{u}}(\boldsymbol{x}) \tag{2.2a}$$

and the singular strain field (2.1b) is regularized as

$$\boldsymbol{\epsilon}(\boldsymbol{x}) = \nabla^{\mathrm{sym}}\bar{\boldsymbol{u}}(\boldsymbol{x}) + \big(\boldsymbol{e} \otimes \boldsymbol{n}\big)^{\mathrm{sym}}\, \Xi_{\mathcal{B}}(\boldsymbol{x}) \tag{2.2b}$$

where the inelastic deformation vector $\boldsymbol{e} := \boldsymbol{w}/b$ is defined as the apparent displacement jump $\boldsymbol{w}$ normalized with respect to the band width b; $\mathscr{H}_{\mathcal{B}}(\boldsymbol{x})$ is a regularized ramp function defined as $\mathscr{H}_{\mathcal{B}}(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \Omega^-$, $\mathscr{H}_{\mathcal{B}}(\boldsymbol{x}) = \frac{1}{b}\big(\boldsymbol{x} - \boldsymbol{x}^*\big) \cdot \boldsymbol{n}$ if $\boldsymbol{x} \in \mathcal{B}$ and $\mathscr{H}_{\mathcal{B}}(\boldsymbol{x}) = 1$ otherwise, with $\boldsymbol{x}^*$ being the spatial coordinates of point $\boldsymbol{x}$ projected along the direction $-\boldsymbol{n}$ to the surface $\mathcal{S}^-$; $\Xi_{\mathcal{B}}(\boldsymbol{x})$ denotes the collocation function within the discontinuity band $\mathcal{B}$, i.e., $\Xi_{\mathcal{B}}(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \mathcal{B}$ and $\Xi_{\mathcal{B}}(\boldsymbol{x}) = 0$ otherwise.

In summary, the strong discontinuity $\mathcal{S}$ induces a discontinuous displacement field $\boldsymbol{u}(\boldsymbol{x})$ and a singular (unbounded) strain field $\boldsymbol{\epsilon}(\boldsymbol{x})$; see Fig. 3(a). Contrariwise, as shown in Fig. 3(b), the kinematic of a regularized discontinuity is characterized by a continuous displacement field $\boldsymbol{u}(\boldsymbol{x})$ and a regular (bounded) strain field $\boldsymbol{\epsilon}(\boldsymbol{x})$.

Remark 2.1 As the discontinuity band width b tends to zero, it follows that

$$\lim_{b\to 0} \mathscr{H}_{\mathcal{B}}(\boldsymbol{x}) = \mathscr{H}_{\mathcal{S}}(\boldsymbol{x}), \qquad \lim_{b\to 0} \frac{1}{b} \varXi_{\mathcal{B}}(\boldsymbol{x}) = \delta_{\mathcal{S}}(\boldsymbol{x}), \qquad \lim_{b\to 0} \boldsymbol{e}\, \varXi_{\mathcal{B}}(\boldsymbol{x}) = \boldsymbol{w} \delta_{\mathcal{S}}(\boldsymbol{x}) \tag{2.3}$$

That is, the strong discontinuity can be regarded as the limit of a regularized one, with a vanishing band width $b \to 0$. Reciprocally, a discontinuity band can be regarded as the convenient regularization of a strong discontinuity. □

Remark 2.2 For either the strong or regularized discontinuity, the strain field $\boldsymbol{\epsilon}(\boldsymbol{x})$ remains continuous across it, i.e.,

$$\boldsymbol{\epsilon}_{\mathcal{S}}^{+} = \boldsymbol{\epsilon}_{\mathcal{S}}^{-} = \boldsymbol{\epsilon}_{\mathcal{S}}^{\pm} \tag{2.4}$$

where $\boldsymbol{\epsilon}_{\mathcal{S}}^{+} := \boldsymbol{\epsilon}(\boldsymbol{x} \in \Omega^{+} \cap \mathcal{S}^{+})$ and $\boldsymbol{\epsilon}_{\mathcal{S}}^{-} := \boldsymbol{\epsilon}(\boldsymbol{x} \in \Omega^{-} \cap \mathcal{S}^{-})$ represent the strains "ahead of" the surface $\mathcal{S}^{+}$ and "behind" the surface $\mathcal{S}^{-}$, respectively. Therefore, in this work no distinction is made between $\boldsymbol{\epsilon}_{\mathcal{S}}^{+}$ and $\boldsymbol{\epsilon}_{\mathcal{S}}^{-}$, and are both denoted by $\boldsymbol{\epsilon}_{\mathcal{S}}^{\pm}$. Contrariwise, the strain $\boldsymbol{\epsilon}_{\mathcal{S}} := \boldsymbol{\epsilon}(\boldsymbol{x} \in \mathcal{S})$ at the discontinuity (band) exhibits a jump with respect to the strain $\boldsymbol{\epsilon}_{\mathcal{S}}^{\pm}$ outside it, which verifies Maxwell's compatibility condition

$$[\![\boldsymbol{\epsilon}]\!] := \boldsymbol{\epsilon}_{\mathcal{S}} - \boldsymbol{\epsilon}_{\mathcal{S}}^{\pm} = \big(\boldsymbol{e} \otimes \boldsymbol{n}\big)^{\text{sym}} = \frac{1}{b}\big(\boldsymbol{w} \otimes \boldsymbol{n}\big)^{\text{sym}} \tag{2.5}$$

The strain jump $[\![\boldsymbol{\epsilon}]\!]$ is inversely proportional to b for a regularized discontinuity (or unbounded for a strong one). □

2.2. *Governing equations*

Restricting the discussion to quasi-static loading, the governing equations of the above problem are expressed as

$$\begin{aligned} &\text{Balance of linear momentum:} && \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b}^{*} = \boldsymbol{0} && \text{in } \Omega \backslash \mathcal{S} \\ &\text{Traction continuity condition:} && \boldsymbol{\sigma}_{\mathcal{S}}^{+} \cdot \boldsymbol{n} = \boldsymbol{\sigma}_{\mathcal{S}}^{-} \cdot \boldsymbol{n} = \boldsymbol{t} && \text{in } \mathcal{S} \end{aligned} \tag{2.6}$$

subjected to the following boundary conditions

$$\begin{aligned} &\text{Traction boundary condition:} && \boldsymbol{\sigma} \cdot \boldsymbol{n}^{*} = \boldsymbol{t}^{*} && \text{in } \varGamma_{t} \\ &\text{Displacement boundary condition:} && \boldsymbol{u} = \boldsymbol{u}^{*} && \text{in } \varGamma_{u} \end{aligned} \tag{2.7}$$

where the second-order tensor $\boldsymbol{\sigma} : \Omega \backslash \mathcal{S} \to \mathbb{R}^{n_{\text{dim}} \times n_{\text{dim}}}$ denotes the stress field in the bulk $\Omega \backslash \mathcal{S} = \Omega^{+} \cup \Omega^{-}$, with $\boldsymbol{\sigma}_{\mathcal{S}}^{+} := \boldsymbol{\sigma}(\boldsymbol{x} \in \Omega^{+} \cap \mathcal{S})$ and $\boldsymbol{\sigma}_{\mathcal{S}}^{-} := \boldsymbol{\sigma}(\boldsymbol{x} \in \Omega^{-} \cap \mathcal{S})$ being the stresses "ahead of" and "behind" the interface $\mathcal{S}$, respectively; the vector $\boldsymbol{t} : \mathcal{S} \to \mathbb{R}^{n_{\text{dim}}}$ represents the cohesive tractions transferred across the discontinuity $\mathcal{S}$.

2.3. *Characteristic angles of the discontinuity (band)*

In the modeling of localized failure in solids, it is necessary to characterize the discontinuity orientation $\boldsymbol{n}$. To this end, let us first consider the following spectral decomposition of the stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \boldsymbol{v}_i \otimes \boldsymbol{v}_i \tag{2.8}$$

where σ_i denote the i-th principal value, with $\boldsymbol{v}_i$ being the corresponding principal vector. Note that the orthogonal property $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \delta_{ij}$ holds for $i, j = 1, 2, 3$.

For general 3D cases, a local orthogonal base $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{p})$ is then introduced, with vectors $\boldsymbol{m}$ and $\boldsymbol{p}$ perpendicular to the normal vector $\boldsymbol{n}$ and tangent to the discontinuity $\mathcal{S}$. In the coordinate system of principal stresses, the base vectors $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{p})$ can be expressed in terms of a set of characteristic angles $\boldsymbol{\theta} := \{\theta_1, \theta_2, \vartheta_1, \vartheta_2\}^{\mathrm{T}}$

$$\boldsymbol{n}(\boldsymbol{\theta}) = \{\sin\vartheta_1 \cos\theta_1,\ \sin\vartheta_1 \sin\theta_1,\ \cos\vartheta_1\}^{\mathrm{T}} \tag{2.9a}$$

$$\boldsymbol{m}(\boldsymbol{\theta}) = \{\sin\vartheta_2 \cos\theta_2,\ \sin\vartheta_2 \sin\theta_2,\ \cos\vartheta_2\}^{\mathrm{T}} \tag{2.9b}$$

$$\boldsymbol{p}(\boldsymbol{\theta}) = \boldsymbol{n}(\boldsymbol{\theta}) \times \boldsymbol{m}(\boldsymbol{\theta}) \tag{2.9c}$$

supplemented with an additional constrained equation

$$C(\boldsymbol{\theta}) := \boldsymbol{n}(\boldsymbol{\theta}) \cdot \boldsymbol{m}(\boldsymbol{\theta}) = \sin\vartheta_1 \sin\vartheta_2 \cos(\theta_1 - \theta_2) + \cos\vartheta_1 \cos\vartheta_2 = 0 \tag{2.9d}$$

where (θ_1, θ_2) and $(\vartheta_1, \vartheta_2)$ denote the spherical azimuth and polar angles, respectively; the operator "$\times$" denotes the Gibbs' vector product (the right hand rule is followed).

In this way, the discontinuity orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\mathrm{cr}})$ can be characterized by a specific set of characteristic angles $\boldsymbol{\theta}^{\mathrm{cr}}$.

3. Discontinuities localized in elastic solids

Once strain localization occurs, it is generally assumed that the bulk material remains elastic during the whole deformation process and all inelastic behavior is localized within the discontinuity (band). This is methodology considered in this section, i.e., inelastic discontinuities localized in elastic solids along preferred orientation.

3.1. General framework

Upon the methodology of inelastic discontinuities localized in elastic solids, the strain field $\boldsymbol{\epsilon}$ is expressed as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{e}} + \boldsymbol{\epsilon}^{\mathrm{in}} \tag{3.1}$$

where $\boldsymbol{\epsilon}^{\mathrm{e}} = \nabla^{\mathrm{sym}} \bar{\boldsymbol{u}}$ represents the elastic strain field; the inelastic strain field $\boldsymbol{\epsilon}^{\mathrm{in}}$ is given by

$$\boldsymbol{\epsilon}^{\mathrm{in}} = (\boldsymbol{w} \otimes \boldsymbol{n})^{\mathrm{sym}}\, \delta_{\mathcal{S}} \qquad \text{or} \qquad \boldsymbol{\epsilon}^{\mathrm{in}} = (\boldsymbol{e} \otimes \boldsymbol{n})^{\mathrm{sym}}\, \Xi_{\mathcal{B}} \tag{3.2}$$

localized within the discontinuity (band).

Without loss of generality, let us consider the following generic constitutive relations

$$\boldsymbol{\sigma} = \mathbb{E}^0 : \boldsymbol{\epsilon}^{\mathrm{e}} = \mathbb{E}^0 : (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\mathrm{in}}), \qquad \boldsymbol{\epsilon}^{\mathrm{e}} = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\mathrm{in}} = \mathbb{C}^0 : \boldsymbol{\sigma} \tag{3.3}$$

or equivalently, in rate forms

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}^0 : \dot{\boldsymbol{\epsilon}}^{\mathrm{e}} = \mathbb{E}^0 : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^{\mathrm{in}}), \qquad \dot{\boldsymbol{\epsilon}}^{\mathrm{e}} = \dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^{\mathrm{in}} = \mathbb{C}^0 : \dot{\boldsymbol{\sigma}} \tag{3.4}$$

where $\mathbb{E}^0$ and $\mathbb{C}^0 := (\mathbb{E}^0)^{-1}$ denote the fourth-order elastic stiffness and compliance tensors, respectively. Note that the constitutive relations (3.3) and (3.4), together with the kinematic decomposition (3.1), hold for both the elastic bulk and the discontinuity (band).

Remark 3.1 As the inelastic strains $\boldsymbol{\epsilon}^{\text{in}}$ are localized completely within the discontinuity (band), the elastic strain field $\boldsymbol{\epsilon}^{\text{e}}$ is continuous in the entire solid. It follows from the constitutive relations (3.3) that, ***though the strain field may be discontinuous or even singular, the resulting stress filed is continuous across the discontinuity (band)***, i.e.,

$$\boxed{\boldsymbol{\sigma}_{\mathcal{S}}^{+} = \boldsymbol{\sigma}_{\mathcal{S}}^{-} = \boldsymbol{\sigma}_{\mathcal{S}}} \tag{3.5}$$

for the stress $\boldsymbol{\sigma}_{\mathcal{S}} := \boldsymbol{\sigma}(\boldsymbol{x} \in \mathcal{S})$ at the discontinuity (band). This conclusion, applicable for both strong and regularized discontinuities, is exactly the *stress continuity condition* upon strain localization in inelastic softening solids [66]; see Section 4.1 for the details. In this case, the traction continuity condition $(2.6)_2$, which describes the relation between the bulk stresses $\boldsymbol{\sigma}_{\mathcal{S}}^{+} = \boldsymbol{\sigma}_{\mathcal{S}}^{-}$ and tractions $\boldsymbol{t}$ transferred by the discontinuity (band), coincides with the classical static constraint at the material level, i.e.,

$$\boldsymbol{t} = \boldsymbol{\sigma}_{\mathcal{S}} \cdot \boldsymbol{n}, \qquad \dot{\boldsymbol{t}} = \dot{\boldsymbol{\sigma}}_{\mathcal{S}} \cdot \boldsymbol{n} \tag{3.6}$$

where the material character $\dot{\boldsymbol{n}} = \boldsymbol{0}$ has been considered. The static constraint is employed in the embedded/smeared crack models [2, 52] and traction-based elastoplastic damage models [67] discussed later in this work. □

3.2. Failure criterion and orientation of the discontinuity (band)

To characterize the discontinuity (band), a *traction-based* failure criterion $\hat{f}(\boldsymbol{t}, q) \le 0$, is introduced. Without loss of generality, let us consider the homogeneous failure function $\hat{f}(\boldsymbol{t}, q)$ of degree $M \ge 1$, i.e.,

$$\hat{f}(\boldsymbol{t}, q) = \frac{1}{M}\big(\partial_{\boldsymbol{t}} \hat{f} \cdot \boldsymbol{t} + \partial_q \hat{f} \cdot q\big) = \frac{1}{M}\big(\hat{\boldsymbol{\gamma}} \cdot \boldsymbol{t} - \hat{h} \cdot q\big) \le 0 \tag{3.7}$$

where $q(\cdot)$ denotes the cohesive strength of the discontinuity (band), dependent on a displacement-like internal variable $\tilde{\kappa}$ (or a strain-like one κ); the derivatives $\hat{\boldsymbol{\gamma}}$ and $\hat{h}$ are expressed as

$$\hat{\boldsymbol{\gamma}} := \frac{\partial \hat{f}}{\partial \boldsymbol{t}}, \qquad \hat{h} := -\frac{\partial \hat{f}}{\partial q} \tag{3.8}$$

As will be clear, the vector $\boldsymbol{\gamma}$ characterizes the evolution laws of the inelastic discontinuity (band).

The above failure criterion $\hat{f}(\boldsymbol{t}, q) \le 0$ is expressed in terms of the traction $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ (hereafter the subscript '$\mathcal{S}$' associated with $\boldsymbol{\sigma}_{\mathcal{S}}$ is dropped for brevity), or equivalently, of the stress tensor $\boldsymbol{\sigma}$ and the normal vector $\boldsymbol{n}$ of the discontinuity (band). Accordingly, the discontinuity orientation $\boldsymbol{n}$ has to be determined, usually in an heuristic manner (e.g., along the major principal stress for mode-I fracture) which might be insufficient for general cases. Owing to the static constraint (3.6), Mohr's maximization postulate [31] is adopted in this work for a given traction-based failure criterion $\hat{f}(\boldsymbol{t}, q) \le 0$. That is, a discontinuity is initiated on the orientation upon which the tractions $\boldsymbol{t}(\boldsymbol{\theta}^{\text{cr}})$ maximize the failure function $\hat{f}\big[\boldsymbol{t}(\boldsymbol{\theta}), q\big]$, i.e.,

$$\hat{\boldsymbol{\theta}}^{\text{cr}} = \arg\max \hat{f}\big[\boldsymbol{t}(\boldsymbol{\theta}), q\big] \qquad \forall \boldsymbol{\theta} \tag{3.9}$$

where the characteristic angles $\widehat{\boldsymbol{\theta}}^{\text{cr}}$ define the discontinuity orientation; the tractions $\boldsymbol{t}(\boldsymbol{\theta})$ acting on the surface with normal vector $\boldsymbol{n}(\boldsymbol{\theta})$ are given by

$$\boldsymbol{t}(\boldsymbol{\theta}) = \boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{\theta}) = t_n \; \boldsymbol{n}(\boldsymbol{\theta}) + t_m \; \boldsymbol{m}(\boldsymbol{\theta}) + t_p \; \boldsymbol{p}(\boldsymbol{\theta}) \tag{3.10}$$

for the local Cartesian components (t_n, t_m, t_p)

$$t_n = \big(\boldsymbol{n} \otimes \boldsymbol{n}\big) : \boldsymbol{\sigma}, \qquad t_m = \big(\boldsymbol{n} \otimes \boldsymbol{m}\big)^{\text{sym}} : \boldsymbol{\sigma}, \qquad t_p = \big(\boldsymbol{n} \otimes \boldsymbol{p}\big)^{\text{sym}} : \boldsymbol{\sigma} \tag{3.11}$$

With the local orthogonal base vectors $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{p})$ introduced in Eqs. (2.9), the traction components (t_n, t_m, t_p) can be expressed in terms of the principal stresses σ_i and the characteristic angles $\boldsymbol{\theta}$.

Accordingly, the following stationarity condition holds

$$\frac{\partial \widehat{f}}{\partial \boldsymbol{\theta}}\bigg|_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} = \left(\widehat{\boldsymbol{\gamma}} \cdot \frac{\partial \boldsymbol{t}}{\partial \boldsymbol{\theta}}\right)_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} = \left(\widehat{\gamma}_n \frac{\partial t_n}{\partial \boldsymbol{\theta}} + \widehat{\gamma}_m \frac{\partial t_m}{\partial \boldsymbol{\theta}} + \widehat{\gamma}_p \frac{\partial t_p}{\partial \boldsymbol{\theta}}\right)_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} = \boldsymbol{0} \tag{3.12}$$

together with a negative definite Hessian matrix

$$\boldsymbol{\theta}^* \cdot \frac{\partial^2 \widehat{f}}{\partial \boldsymbol{\theta}^2}\bigg|_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} \cdot \boldsymbol{\theta}^* < 0 \qquad \forall \boldsymbol{\theta}^* \tag{3.13}$$

where the local Cartesian components $(\widehat{\gamma}_n, \widehat{\gamma}_m, \widehat{\gamma}_p)$ of the vector $\widehat{\boldsymbol{\gamma}}$ are expressed as

$$\widehat{\gamma}_n = \frac{\partial \widehat{f}}{\partial t_n}, \qquad \widehat{\gamma}_m = \frac{\partial \widehat{f}}{\partial t_m}, \qquad \widehat{\gamma}_p = \frac{\partial \widehat{f}}{\partial t_p} \tag{3.14}$$

Recalling the traction components (3.11), the stationarity condition (3.12) results in a set of nonlinear equations so that the discontinuity angles $\widehat{\boldsymbol{\theta}}^{\text{cr}}$ can be solved. If the solution exists, it depends only on the given traction-based failure criterion and the stress state. Note that the constraint (2.9d) can be handled by the Lagrangian multiplier method.

3.3. *Strong/regularized discontinuity approaches*

With respect to the strategies dealing with the kinematic unknowns $\boldsymbol{w}$ (or $\boldsymbol{e} := \boldsymbol{w}/b$), two approaches can be identified from the methodology of inelastic discontinuities localized in elastic solids. Either the singular kinematics (2.1) or the regularized one (2.2) can be incorporated in both approaches.

Let us first consider the *strong/regularized discontinuity approaches* in which the kinematic unknowns $\boldsymbol{w}$ (or $\boldsymbol{e}$) associated to the discontinuity (band) are retained as independent variables. With the singular kinematics (2.1), the governing equations (2.6) and the traction boundary condition $(2.7)_1$ can be cast as: Find $\boldsymbol{u} \in \mathscr{U}$ such that [29]

$$\int_{\Omega} \nabla^{\text{sym}} \delta \bar{\boldsymbol{u}} : \boldsymbol{\sigma} \; \mathrm{d}\Omega + \int_{\mathcal{S}} \delta \boldsymbol{w} \cdot \boldsymbol{t} \; \mathrm{d}\mathcal{S} = \int_{\Omega} \delta \boldsymbol{u} \cdot \boldsymbol{b}^* \; \mathrm{d}\Omega + \int_{\Gamma_t} \delta \boldsymbol{u} \cdot \boldsymbol{t}^* \; \mathrm{d}\Gamma \qquad \forall \delta \boldsymbol{u} \in \mathcal{V} \tag{3.15}$$

where the trial function $\boldsymbol{u} \in \mathscr{U}$ and its variation $\delta \boldsymbol{u} \in \mathcal{V}$ are chosen from the spaces

$$\mathscr{U} := \big\{\boldsymbol{u} \big| \boldsymbol{u} = \bar{\boldsymbol{u}} + \mathscr{H}_{\mathcal{S}} \; \hat{\boldsymbol{u}}; \; \boldsymbol{u} = \boldsymbol{u}^* \text{ in } \Gamma_u; \; \hat{\boldsymbol{u}} = \boldsymbol{w} \text{ in } \mathcal{S}\big\} \tag{3.16a}$$

$$\mathcal{V} := \big\{\delta \boldsymbol{u} \big| \delta \boldsymbol{u} = \delta \bar{\boldsymbol{u}} + \mathscr{H}_{\mathcal{S}} \; \delta \hat{\boldsymbol{u}}; \; \delta \boldsymbol{u} = \boldsymbol{0} \text{ in } \Gamma_u; \; \delta \hat{\boldsymbol{u}} = \delta \boldsymbol{w} \text{ in } \mathcal{S}\big\} \tag{3.16b}$$

Alternatively, if the regularized kinematics (2.2) is employed, the weak form (3.15) becomes: Find $\boldsymbol{u} \in \mathscr{U}$ such that

$$\int_{\Omega} \nabla^{\mathrm{sym}} \delta \bar{\boldsymbol{u}} : \boldsymbol{\sigma} \, \mathrm{d}\Omega + \int_{\mathcal{B}} \delta \boldsymbol{e} \cdot \boldsymbol{t} \, \mathrm{d}\mathcal{B} = \int_{\Omega} \delta \boldsymbol{u} \cdot \boldsymbol{b}^* \, \mathrm{d}\Omega + \int_{\Gamma_t} \delta \boldsymbol{u} \cdot \boldsymbol{t}^* \, \mathrm{d}\Gamma \qquad \forall \delta \boldsymbol{u} \in \mathcal{V} \tag{3.17}$$

where the trial function $\boldsymbol{u} \in \mathscr{U}$ and its variation $\delta \boldsymbol{u} \in \mathcal{V}$ are chosen within the spaces

$$\mathscr{U} := \left\{ \boldsymbol{u} \middle| \boldsymbol{u} = \bar{\boldsymbol{u}} + \mathscr{H}_{\mathcal{B}} \, \hat{\boldsymbol{u}}; \; \boldsymbol{u} = \boldsymbol{u}^* \text{ in } \Gamma_u; \; \hat{\boldsymbol{u}} = \boldsymbol{w} \text{ in } \mathcal{S} \right\} \tag{3.18a}$$

$$\mathcal{V} := \left\{ \delta \boldsymbol{u} \middle| \delta \boldsymbol{u} = \delta \bar{\boldsymbol{u}} + \mathscr{H}_{\mathcal{B}} \, \delta \hat{\boldsymbol{u}}; \; \delta \boldsymbol{u} = \boldsymbol{0} \text{ in } \Gamma_u; \; \delta \hat{\boldsymbol{u}} = \delta \boldsymbol{w} \text{ in } \mathcal{S} \right\} \tag{3.18b}$$

In the weak form (3.15) or (3.17), the displacement boundary condition $(2.7)_2$ are enforced in strong form, and the variations $(\delta \bar{\boldsymbol{u}}, \delta \hat{\boldsymbol{u}})$ are of sufficient regularity.

To proceed, the above weak form (3.15) or (3.17) has to be supplemented with constitutive relations for both the bulk stress $\boldsymbol{\sigma}$ and tractions $\boldsymbol{t}$ transferred across the discontinuity (band). The linear elastic relation (3.3) is assumed for the bulk. As the tractions $\boldsymbol{t}$ are work conjugated to the displacement jumps $\boldsymbol{w}$, a cohesive model in terms of tractions vs. displacement jumps (or inelastic deformations) should be fed to the discontinuity (band).

Within the framework of irreversible thermodynamics, thermodynamically consistent cohesive models for the strong/regularized discontinuities can be developed; see Wu and Cervera [66] for the details. Here, only the necessary formulas are given and all the derivations are omitted.

3.3.1. *Cohesive model for the discontinuity*

The discontinuity (band) induces both stiffness degradation and irreversible deformations. To account for them, the displacement jump $\boldsymbol{w}$ and the resulting singular inelastic strain $\boldsymbol{\epsilon}^{\mathrm{in}}$ are decomposed as

$$\boldsymbol{w} = \boldsymbol{w}^{\mathrm{d}} + \boldsymbol{w}^{\mathrm{p}}, \qquad \boldsymbol{\epsilon}^{\mathrm{in}} = \boldsymbol{\epsilon}^{\mathrm{d}} + \boldsymbol{\epsilon}^{\mathrm{p}} \tag{3.19}$$

where $(\boldsymbol{w}^{\mathrm{d}}, \boldsymbol{w}^{\mathrm{p}})$ represent the recoverable/unrecoverable parts of the displacement jump $\boldsymbol{w}$; the resulting damage (recoverable) strain $\boldsymbol{\epsilon}^{\mathrm{d}} = \left(\boldsymbol{w}^{\mathrm{d}} \otimes \boldsymbol{n}\right)^{\mathrm{sym}} \delta_{\mathcal{S}}$ and plastic (unrecoverable) strain $\boldsymbol{\epsilon}^{\mathrm{p}} = \left(\boldsymbol{w}^{\mathrm{p}} \otimes \boldsymbol{n}\right)^{\mathrm{sym}} \delta_{\mathcal{S}}$ are both singular.

Accordingly, the strong discontinuity can be described by the following plastic-damage type cohesive relations

$$\boldsymbol{t} = \widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot \boldsymbol{w}^{\mathrm{d}} = \widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot \left(\boldsymbol{w} - \boldsymbol{w}^{\mathrm{p}}\right), \qquad \boldsymbol{w}^{\mathrm{d}} = \boldsymbol{w} - \boldsymbol{w}^{\mathrm{p}} = \widetilde{\boldsymbol{C}}^{\mathrm{d}} \cdot \boldsymbol{t} \tag{3.20}$$

or the rate forms

$$\dot{\boldsymbol{t}} = \widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot \left(\dot{\boldsymbol{w}} - \dot{\boldsymbol{w}}^{\mathrm{dis}}\right), \qquad \dot{\boldsymbol{w}} = \widetilde{\boldsymbol{C}}^{\mathrm{d}} \cdot \dot{\boldsymbol{t}} + \dot{\boldsymbol{w}}^{\mathrm{dis}} \tag{3.21}$$

where the second-order tensors $\widetilde{\boldsymbol{E}}^{\mathrm{d}}$ and $\widetilde{\boldsymbol{C}}^{\mathrm{d}} := \left(\widetilde{\boldsymbol{E}}^{\mathrm{d}}\right)^{-1}$ denote the (variable) discontinuity compliance tensors, respectively; $\dot{\boldsymbol{w}}^{\mathrm{dis}}$ represents the *dissipative displacement jump rate* meaningful only in rate sense, defined as

$$\dot{\boldsymbol{w}}^{\mathrm{dis}} = \dot{\widetilde{\boldsymbol{C}}}^{\mathrm{d}} \cdot \boldsymbol{t} + \dot{\boldsymbol{w}}^{\mathrm{p}} \tag{3.22}$$

with $\dot{\widetilde{\boldsymbol{C}}}^{\mathrm{d}} \cdot \boldsymbol{t}$ and $\dot{\boldsymbol{w}}^{\mathrm{p}}$ being its damage and plastic components; see Fig. 4.

In the above cohesive relations, the compliance $\widetilde{\boldsymbol{C}}^{\mathrm{d}}$ (or the stiffness $\widetilde{\boldsymbol{E}}^{\mathrm{d}}$), the unrecoverable displacement jump $\boldsymbol{w}^{\mathrm{p}}$ and the displacement-like variable $\tilde{\kappa}$ are all internal variables. In analogy to the classical plasticity theory, the associated evolution laws can be derived from the postulate of maximum energy dissipation [56], i.e.,

$$\dot{\boldsymbol{w}}^{\mathrm{dis}} = \dot{\widetilde{\boldsymbol{C}}}^{\mathrm{d}} \cdot \boldsymbol{t} + \dot{\boldsymbol{w}}^{\mathrm{p}} = \tilde{\lambda} \boldsymbol{\gamma}, \qquad \dot{\tilde{\kappa}} = \tilde{\lambda} h \tag{3.23}$$

where the localized Lagrangian multiplier $\tilde{\lambda}$ satisfies the following Kuhn-Tucker loading/unloading rules

$$\tilde{\lambda} \geq 0, \qquad \hat{f} \leq 0, \qquad \tilde{\lambda} \hat{f} = 0 \tag{3.24}$$

for the dissipative flow vector $\boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}}$ and softening function $h = \hat{h}$ given in Eq. (3.8).

To discriminate between damage and plastic contributions to the dissipative displacement jump rate $\dot{\boldsymbol{w}}^{\mathrm{dis}}$, a material parameter $\xi \in [0, 1]$ is introduced so that

$$\dot{\boldsymbol{w}}^{\mathrm{p}} = \big(1 - \xi\big)\, \dot{\boldsymbol{w}}^{\mathrm{dis}} = \big(1 - \xi\big)\, \tilde{\lambda} \boldsymbol{\gamma} \tag{3.25a}$$

$$\dot{\widetilde{\boldsymbol{C}}}^{\mathrm{d}} \cdot \boldsymbol{t} = \xi\, \dot{\boldsymbol{w}}^{\mathrm{dis}} = \xi\, \tilde{\lambda} \boldsymbol{\gamma} \tag{3.25b}$$

The postulate of maximum energy dissipation determines only the components of the compliance $\widetilde{\boldsymbol{C}}^{\mathrm{d}}$ appearing in the product $\boldsymbol{t} \cdot \dot{\widetilde{\boldsymbol{C}}}^{\mathrm{d}} \cdot \boldsymbol{t}$, leaving the remaining ones undefined. A particular evolution law satisfying Eq. (3.25b) is given by

$$\dot{\widetilde{\boldsymbol{C}}}^{\mathrm{d}} = \xi\, \tilde{\lambda} \frac{\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}}{\boldsymbol{\gamma} \cdot \boldsymbol{t}} \tag{3.26}$$

as long as the denominator $\boldsymbol{\gamma} \cdot \boldsymbol{t} \neq 0$. The cases $\xi = 0$ and $\xi = 1$ correspond to cohesive models of pure plastic and pure damage types, respectively. For the intermediate value $\xi \in (0, 1)$, the discontinuity compliance $\widetilde{\boldsymbol{C}}^{\mathrm{d}}$ and the plastic displacement jump $\boldsymbol{w}^{\mathrm{p}}$ are both internal variables and a combined plastic-damage cohesive model follows.

When the discontinuity is inactive, i.e., $\hat{f}(\boldsymbol{t}, q) < 0$, it follows from Eq. (3.24) that $\tilde{\lambda} = 0$; when the discontinuity evolves, it follows that $\hat{f}(\boldsymbol{t}, q) = 0$ and the localized Lagrangian multiplier $\tilde{\lambda} > 0$ is solved from the consistency condition $\dot{\hat{f}} = 0$, i.e.,

$$\tilde{\lambda} = \frac{\boldsymbol{\gamma} \cdot \widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot \dot{\boldsymbol{w}}}{\boldsymbol{\gamma} \cdot \widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot \boldsymbol{\gamma} + h \cdot \widetilde{H} \cdot h} = \frac{\boldsymbol{\gamma} \cdot \dot{\boldsymbol{t}}}{h \cdot \widetilde{H} \cdot h} \tag{3.27}$$

for the softening modulus $\widetilde{H} := \partial q / \partial \tilde{\kappa} < 0$.

Combination of Eqs. (3.21) and (3.23) then yields the following rate constitutive relations

$$\dot{\boldsymbol{t}} = \widetilde{\boldsymbol{E}} \cdot \big(\dot{\boldsymbol{w}} - \tilde{\lambda} \boldsymbol{\gamma}\big) = \widetilde{\boldsymbol{E}}^{\mathrm{d}}_{\tan} \cdot \dot{\boldsymbol{w}} \tag{3.28a}$$

$$\dot{\boldsymbol{w}} = \widetilde{\boldsymbol{C}} \cdot \boldsymbol{t} + \tilde{\lambda} \boldsymbol{\gamma} = \widetilde{\boldsymbol{C}}^{\mathrm{d}}_{\tan} \cdot \dot{\boldsymbol{t}} \tag{3.28b}$$

where the tangent stiffness and compliance are expressed as

$$\widetilde{\boldsymbol{E}}^{\mathrm{d}}_{\tan} = \widetilde{\boldsymbol{E}}^{\mathrm{d}} - \frac{\widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot (\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}) \cdot \widetilde{\boldsymbol{E}}^{\mathrm{d}}}{\boldsymbol{\gamma} \cdot \widetilde{\boldsymbol{E}}^{\mathrm{d}} \cdot \boldsymbol{\gamma} + h \cdot \widetilde{H} \cdot h} \tag{3.29a}$$

$$\widetilde{\boldsymbol{C}}^{\mathrm{d}}_{\tan} = \widetilde{\boldsymbol{C}}^{\mathrm{d}} + \frac{\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}}{h \cdot \widetilde{H} \cdot h} \tag{3.29b}$$

both being symmetric due to the associated evolution laws adopted.

3.3.2. Regularized cohesive model for the discontinuity band

Correspondingly to the decomposition (3.19), the inelastic deformation vector $\boldsymbol{e} := \boldsymbol{w}/b$ and the resulting inelastic strain $\boldsymbol{\epsilon}^{\text{in}}$ admit similar additive forms

$$\boldsymbol{e} = \boldsymbol{e}^{\text{d}} + \boldsymbol{e}^{\text{p}}, \qquad \boldsymbol{\epsilon}^{\text{in}} = \boldsymbol{\epsilon}^{\text{d}} + \boldsymbol{\epsilon}^{\text{p}} \tag{3.30}$$

where the damage and plastic strains $(\boldsymbol{\epsilon}^{\text{d}}, \boldsymbol{\epsilon}^{\text{p}})$, localized within the discontinuity band, are expressed as

$$\boldsymbol{\epsilon}^{\text{d}} = \left(\boldsymbol{e}^{\text{d}} \otimes \boldsymbol{n}\right)^{\text{sym}} \varXi_{\mathcal{B}}, \qquad \boldsymbol{\epsilon}^{\text{p}} = \left(\boldsymbol{e}^{\text{p}} \otimes \boldsymbol{n}\right)^{\text{sym}} \varXi_{\mathcal{B}} \tag{3.31}$$

for the recoverable and unrecoverable deformation vectors $\boldsymbol{e}^{\text{d}} := \boldsymbol{w}^{\text{d}}/b$ and $\boldsymbol{e}^{\text{p}} := \boldsymbol{w}^{\text{p}}/b$, respectively.

Similarly to the strong discontinuity, the regularized cohesive relations are expressed as

$$\boldsymbol{t} = \boldsymbol{E}^{\text{d}} \cdot \boldsymbol{e}^{\text{d}} = \boldsymbol{E}^{\text{d}} \cdot \left(\boldsymbol{e} - \boldsymbol{e}^{\text{p}}\right), \qquad \boldsymbol{e}^{\text{d}} = \boldsymbol{e} - \boldsymbol{e}^{\text{p}} = \boldsymbol{C}^{\text{d}} \cdot \boldsymbol{t} \tag{3.32}$$

or the rate forms

$$\dot{\boldsymbol{t}} = \boldsymbol{E}^{\text{d}} \cdot \left(\dot{\boldsymbol{e}} - \dot{\boldsymbol{e}}^{\text{dis}}\right), \qquad \dot{\boldsymbol{e}} = \boldsymbol{C}^{\text{d}} \cdot \dot{\boldsymbol{t}} + \dot{\boldsymbol{e}}^{\text{dis}} \tag{3.33}$$

where the second-order tensors $\boldsymbol{E}^{\text{d}}$ and $\boldsymbol{C}^{\text{d}} = \left(\boldsymbol{E}^{\text{d}}\right)^{-1}$ represent the stiffness and compliance of the discontinuity band, respectively; the *dissipative deformation rate* $\dot{\boldsymbol{e}}^{\text{dis}}$ is defined as

$$\dot{\boldsymbol{e}}^{\text{dis}} = \dot{\boldsymbol{C}}^{\text{d}} \cdot \boldsymbol{t} + \dot{\boldsymbol{e}}^{\text{p}} \tag{3.34}$$

with $\dot{\boldsymbol{C}}^{\text{d}} \cdot \boldsymbol{t}$ and $\dot{\boldsymbol{e}}^{\text{p}}$ being its damage and plastic components, respectively; see Fig. 5.

The associated evolution laws for the internal variables $\left\{\boldsymbol{C}^{\text{d}}, \boldsymbol{e}^{\text{p}}, \kappa\right\}$ are similarly derived as

$$\dot{\boldsymbol{e}}^{\text{dis}} = \dot{\lambda}\boldsymbol{\gamma}, \qquad \dot{\kappa} = \lambda h \tag{3.35}$$

Furthermore, a material parameter $\xi \in [0, 1]$ is introduced so that

$$\dot{\boldsymbol{e}}^{\text{p}} = (1-\xi)\, \dot{\boldsymbol{e}}^{\text{dis}} = (1-\xi)\, \lambda\boldsymbol{\gamma} \tag{3.36a}$$

$$\dot{\boldsymbol{C}}^{\text{d}} \cdot \boldsymbol{t} = \xi\, \dot{\boldsymbol{e}}^{\text{dis}} = \xi\, \lambda\boldsymbol{\gamma} \qquad \Longrightarrow \qquad \dot{\boldsymbol{C}}^{\text{d}} = \xi\, \lambda \frac{\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}}{\boldsymbol{\gamma} \cdot \boldsymbol{t}} \tag{3.36b}$$

where the dissipative flow vector $\boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}}$ and softening function $h = \widehat{h}$ are also given in Eq. (3.8); the regularized Lagrangian multiplier λ satisfies the Kuhn-Tucker loading/unloading rules

$$\lambda \geq 0, \qquad \widehat{f} \leq 0, \qquad \lambda \widehat{f} = 0 \tag{3.37}$$

For an active discontinuity band, the regularized Lagrangian multiplier $\lambda > 0$ is solved from $\dot{\widehat{f}}(\boldsymbol{t}, q) = 0$ as

$$\lambda = \frac{\boldsymbol{\gamma} \cdot \boldsymbol{E}^{\text{d}} \cdot \dot{\boldsymbol{e}}}{\boldsymbol{\gamma} \cdot \boldsymbol{E}^{\text{d}} \cdot \boldsymbol{\gamma} + h \cdot H \cdot h} = \frac{\boldsymbol{\gamma} \cdot \dot{\boldsymbol{t}}}{h \cdot H \cdot h} \tag{3.38}$$

for the strain-driven softening modulus $H := \partial q/\partial \kappa < 0$.

Therefore, the rate constitutive relations for the regularized discontinuity are expressed as

$$\dot{\boldsymbol{t}} = \boldsymbol{E}^{\mathrm{d}} \cdot \big(\dot{\boldsymbol{e}} - \lambda \boldsymbol{\gamma}\big) = \boldsymbol{E}^{\mathrm{d}}_{\tan} \cdot \dot{\boldsymbol{e}} \tag{3.39a}$$

$$\dot{\boldsymbol{e}} = \boldsymbol{C}^{\mathrm{d}} \cdot \dot{\boldsymbol{t}} + \lambda \boldsymbol{\gamma} = \boldsymbol{C}^{\mathrm{d}}_{\tan} \cdot \dot{\boldsymbol{t}} \tag{3.39b}$$

with the following tangent stiffness and compliance

$$\boldsymbol{E}^{\mathrm{d}}_{\tan} = \boldsymbol{E}^{\mathrm{d}} - \frac{\boldsymbol{E}^{\mathrm{d}} \cdot (\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}) \cdot \boldsymbol{E}^{\mathrm{d}}}{\boldsymbol{\gamma} \cdot \boldsymbol{E}^{\mathrm{d}} \cdot \boldsymbol{\gamma} + h \cdot H \cdot h} \tag{3.40a}$$

$$\boldsymbol{C}^{\mathrm{d}}_{\tan} = \boldsymbol{C}^{\mathrm{d}} + \frac{\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}}{h \cdot H \cdot h} \tag{3.40b}$$

As expected, the above tangent tensors are also symmetric due to the associated evolution laws adopted.

3.3.3. Fracture energy and the equivalence conditions

For a strong discontinuity, the external energy supplied to the solid during whole failure process is evaluated as [66]

$$W = \int_{\mathcal{S}} \int_0^\infty \Big(\frac{1}{2}\, \boldsymbol{t} \cdot \dot{\tilde{\boldsymbol{C}}}^{\mathrm{d}} \cdot \boldsymbol{t} + \boldsymbol{t} \cdot \dot{\boldsymbol{w}}^{\mathrm{p}}\Big)\, \mathrm{d}T \; \mathrm{d}\mathcal{S} \tag{3.41}$$

for the time $T \in [0, \infty)$. Substitution of the evolution laws (3.23)$_2$ and (3.25) yields the fracture energy G_{f}, usually regarded as a material property and defined as energy dissipation per unit of discontinuity area $A_{\mathcal{S}}$, i.e.,

$$G_{\mathrm{f}} := \frac{W}{A_{\mathcal{S}}} = \Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty \tilde{\lambda} \boldsymbol{\gamma} \; \mathrm{d}T = \Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty q(\tilde{\kappa}) \; \mathrm{d}\tilde{\kappa} \tag{3.42}$$

where the relation $\widehat{\boldsymbol{\gamma}} \cdot \boldsymbol{t} = \widehat{h} \cdot q$ for an active discontinuity (band) has been considered. Once the displacement-driven softening law $q(\tilde{\kappa})$ is given, the fracture energy G_{f} can be evaluated explicitly from Eq. (3.42). In other words, the parameters involved in the softening law $q(\tilde{\kappa})$ can be determined for the given fracture energy G_{f}.

Similarly, for the regularized discontinuity it follows that

$$W = \int_{\mathcal{B}} \int_0^\infty \Big(\frac{1}{2}\, \boldsymbol{t} \cdot \dot{\boldsymbol{C}}^{\mathrm{d}} \cdot \boldsymbol{t} + \boldsymbol{t} \cdot \dot{\boldsymbol{e}}^{\mathrm{p}}\Big)\, \mathrm{d}T \; \mathrm{d}\mathcal{B} \tag{3.43}$$

Calling for the evolution laws (3.36), the fracture energy G_{f} is evaluated as

$$G_{\mathrm{f}} := \frac{W}{A_{\mathcal{S}}} = \frac{W}{V_{\mathcal{B}}/b} = b\Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty \lambda \boldsymbol{\gamma} \; \mathrm{d}T = b\Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty q(\kappa) \; \mathrm{d}\kappa \tag{3.44}$$

for the volume $V_{\mathcal{B}} = b A_{\mathcal{S}}$ of the discontinuity band. It can be concluded that for the given fracture energy G_{f}, the softening function $q(\kappa)$ also depends on the band width b.

The fracture energy (3.44) allows introducing the so-called specific fracture energy g_{f}, i.e., the dissipation per unit volume of the discontinuity band $V_{\mathcal{B}}$, in a straightforward manner

$$g_{\mathrm{f}} := \frac{W}{V_{\mathcal{B}}} = \int_0^\infty \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \; \mathrm{d}T = \frac{1}{b} G_{\mathrm{f}} \tag{3.45}$$

This is exactly the relation between the crack band theory [5] and the fictitious crack model [22].

For the case of a vanishing band width $b \to 0$, the relation $g_{\text{f}} = G_{\text{f}}\, \delta_{\mathcal{S}}$ holds. It then follows that

$$W = \int_{\mathcal{B}} g_{\text{f}} \, \mathrm{d}\mathcal{B} = \int_{\mathcal{B}} G_{\text{f}}\, \delta_{\mathcal{S}} \, \mathrm{d}\mathcal{B} = \int_{\mathcal{S}} G_{\text{f}} \, \mathrm{d}\mathcal{S} \tag{3.46}$$

which recovers the classical definition of fracture energy.

For the same traction-based failure criterion $\hat{f}(\boldsymbol{t}, q) \leq 0$ and equivalent softening laws $q(\kappa) = q(\tilde{\kappa})$, the results (3.42) and (3.44) imply that identical fracture energies G_{f} can be obtained for both models, providing that

$$\kappa = \frac{1}{b}\tilde{\kappa}, \qquad \dot{\kappa} = \frac{1}{b}\dot{\tilde{\kappa}} \qquad \Longleftrightarrow \qquad \lambda = \frac{1}{b}\tilde{\lambda}, \qquad \frac{1}{H} = \frac{1}{b}\frac{1}{\tilde{H}} \tag{3.47}$$

Upon satisfying the above conditions, it follows from the damage evolution laws (3.26) and (3.36b) that

$$\boldsymbol{C}^{\mathrm{d}} = \frac{1}{b}\tilde{\boldsymbol{C}}^{\mathrm{d}}, \qquad \tilde{\boldsymbol{E}}^{\mathrm{d}} = \frac{1}{b}\boldsymbol{E}^{\mathrm{d}} \tag{3.48}$$

That is, the strong discontinuity transforms into an equivalent regularized counterpart with a finite band width $b \not\to 0$.

Contrariwise, for the regularized discontinuity with a vanishing band width $b \to 0$, the conditions (3.47) become

$$\kappa = \tilde{\kappa}\delta_{\mathcal{S}}, \qquad \dot{\kappa} = \dot{\tilde{\kappa}}\delta_{\mathcal{S}} \qquad \Longleftrightarrow \qquad \lambda = \tilde{\lambda}\delta_{\mathcal{S}}, \qquad \frac{1}{H} = \frac{1}{\tilde{H}}\delta_{\mathcal{S}} \tag{3.49}$$

so that the following relations hold

$$\boldsymbol{C}^{\mathrm{d}} = \tilde{\boldsymbol{C}}^{\mathrm{d}}\delta_{\mathcal{S}}, \qquad \tilde{\boldsymbol{E}}^{\mathrm{d}} = \boldsymbol{E}^{\mathrm{d}}\delta_{\mathcal{S}} \tag{3.50}$$

In this case, the regularized discontinuity localizes into an equivalent strong discontinuity.

As can be seen, all the kinematic variables are inversely proportional to the band width b for the regularized discontinuity or even singular for a strong one. ***The above results, together with kinematic equivalence stated in Remark 2.1, clearly show that the strong and regularized discontinuity approaches are consistently related.***

3.4. Embedded/smeared discontinuity approaches

Providing the cohesive model for the discontinuity (band) is known, e.g., as developed in Section 3.3.1 or 3.3.2, it is possible to establish alternative embedded/smeared discontinuity approaches.

In such approaches, the weak form of the governing equation $(2.6)_1$ and the traction boundary condition $(2.7)_1$ can be re-stated as: Find $\boldsymbol{u} \in \mathscr{U}$ such that

$$\int_{\Omega} \nabla^{\mathrm{sym}} \delta\boldsymbol{u} : \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \delta\boldsymbol{u} \cdot \boldsymbol{b}^* \, \mathrm{d}\Omega + \int_{\Gamma_t} \delta\boldsymbol{u} \cdot \boldsymbol{t}^* \, \mathrm{d}\Gamma \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} \tag{3.51}$$

with the stress $\boldsymbol{\sigma}_{\mathcal{S}}$ at the discontinuity (band) determined from an inelastic material model satisfying the static constraint (3.6). Here, $\mathscr{U}$ is the trial space introduced in Eq. (3.16a) for a strong discontinuity or in Eq. (3.18a) for a regularized one; the standard test space $\mathcal{V}$ is defined as

$$\mathcal{V} := \left\{ \delta\boldsymbol{u} \text{ of sufficient regularity} \middle| \delta\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_u \right\} \tag{3.52}$$

As can be seen, the traction continuity condition (2.6)$_2$ and the displacement jump $\boldsymbol{w}$ (or the inelastic deformation vector $\boldsymbol{e}$) are not explicitly considered in the weak form (3.51). But rather, they are accounted for indirectly at the local (material constitutive) level through the inelastic material models fulfilling the classical static constraint (3.6). Owing to the equivalence shown in Section 3.3.3, only the smeared discontinuity model employing the regularized kinematics (2.2) is addressed. Its correspondence to the embedded counterpart is briefly discussed in Remark 3.2.

To derive the smeared discontinuity model, let us recall the constitutive relations (3.4) and (3.39a). It follows from the static constraint (3.6)$_2$ that

$$\boldsymbol{E}_{\tan}^{\mathrm{d}} \cdot \dot{\boldsymbol{e}} = \mathbb{E}^0 : \Big[\dot{\boldsymbol{\epsilon}} - \big(\dot{\boldsymbol{e}} \otimes \boldsymbol{n}\big)^{\mathrm{sym}}\Big] \cdot \boldsymbol{n} = \big(\mathbb{E}^0 : \dot{\boldsymbol{\epsilon}}\big) \cdot \boldsymbol{n} - \big(\boldsymbol{n} \cdot \mathbb{E}^0 \cdot \boldsymbol{n}\big) \cdot \dot{\boldsymbol{e}} \tag{3.53}$$

Accordingly, the inelastic deformation vector rate $\dot{\boldsymbol{e}}$ is determined in terms of the strain rate $\dot{\boldsymbol{\epsilon}}$ as

$$\dot{\boldsymbol{e}} = \big(\boldsymbol{E}_{\tan}^{\mathrm{d}} + \boldsymbol{n} \cdot \mathbb{E}^0 \cdot \boldsymbol{n}\big)^{-1} \cdot \big(\mathbb{E}^0 : \dot{\boldsymbol{\epsilon}}\big) \cdot \boldsymbol{n} \tag{3.54}$$

Therefore, the stress rate $\dot{\boldsymbol{\sigma}}$ is given from Eq. (3.4), i.e.,

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}^0 : \Big[\dot{\boldsymbol{\epsilon}} - \big(\dot{\boldsymbol{e}} \otimes \boldsymbol{n}\big)^{\mathrm{sym}}\Big] = \mathbb{E}_{\tan} : \dot{\boldsymbol{\epsilon}} \tag{3.55}$$

where the tangent stiffness tensor $\mathbb{E}_{\tan}$ is expressed as

$$\mathbb{E}_{\tan} = \mathbb{E}^0 - \mathbb{E}^0 : \Big[\big(\boldsymbol{E}_{\tan}^{\mathrm{d}} + \boldsymbol{n} \cdot \mathbb{E}^0 \cdot \boldsymbol{n}\big)^{-1} \overline{\underline{\otimes}} \boldsymbol{N}\Big]^{\mathrm{sym}} : \mathbb{E}^0 \tag{3.56}$$

with a second-order geometric tensor $\boldsymbol{N} := \boldsymbol{n} \otimes \boldsymbol{n}$.

Alternatively, it follows from the relation (3.39b) and the static constraint (3.6)$_2$ that

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{in}} = \big(\dot{\boldsymbol{e}} \otimes \boldsymbol{n}\big)^{\mathrm{sym}} = \Big[\big(\boldsymbol{C}_{\tan}^{\mathrm{d}} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\big) \otimes \boldsymbol{n}\Big]^{\mathrm{sym}} = \big(\boldsymbol{C}_{\tan}^{\mathrm{d}} \overline{\underline{\otimes}} \boldsymbol{N}\big)^{\mathrm{sym}} : \dot{\boldsymbol{\sigma}} \tag{3.57}$$

The strain rate $\dot{\boldsymbol{\epsilon}}$ is given from Eq. (3.1)

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\epsilon}}^{\mathrm{in}} = \mathbb{C}_{\tan} : \dot{\boldsymbol{\sigma}} \tag{3.58}$$

for the tangent compliance $\mathbb{C}_{\tan}$ expressed as

$$\mathbb{C}_{\tan} = \mathbb{C}^0 + \big(\boldsymbol{C}_{\tan}^{\mathrm{d}} \overline{\underline{\otimes}} \boldsymbol{N}\big)^{\mathrm{sym}} \tag{3.59}$$

Note that the tangent compliance $\mathbb{C}_{\tan}$ can also be directly derived from Eq. (3.56) through the Sherman-Morrison-Woodburg formula [18].

The above constitutive relations correspond exactly to those in the smeared crack model [51–53], as an extension of the original form proposed by Rashid [48]. As can be seen, owning to stress continuity (3.5) upon strain localization, the kinematic unknowns $\boldsymbol{w}$ (or $\boldsymbol{e} := \boldsymbol{w}/b$) can be eliminated at the local material-point level through the static constraint (3.6), resulting in a single constitutive model to describe the overall inelastic behavior of the solid.

Remark 3.2 If the singular kinematics (2.1) is adopted, constitutive relations similarly to Eqs. (3.55) and (3.58) can be obtained for the embedded discontinuity model, after accounting for the relations (3.50). For instance, the inelastic strain rate $\dot{\boldsymbol{\epsilon}}^{\text{in}}$ is expressed as

$$\dot{\boldsymbol{\epsilon}}^{\text{in}} = \left(\widetilde{\boldsymbol{C}}_{\text{tan}}^{\text{d}} \,\overline{\underline{\otimes}}\, \boldsymbol{N}\right)^{\text{sym}} \delta_{\mathcal{S}} : \dot{\boldsymbol{\sigma}} \tag{3.60}$$

so that the tangent compliance in Eq. (3.58) becomes

$$\mathbb{C}_{\text{tan}} = \mathbb{C}^0 + \left(\widetilde{\boldsymbol{C}}_{\text{tan}}^{\text{d}} \,\overline{\underline{\otimes}}\, \boldsymbol{N}\right)^{\text{sym}} \delta_{\mathcal{S}} \tag{3.61}$$

Other correspondences can be similarly established; the details are omitted here. □

Remark 3.3 Calling for the decomposition (3.19) or (3.30), the strain $\boldsymbol{\epsilon}$ at the discontinuity (band) is additively split into its elastic, damage and plastic parts

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\text{e}} + \boldsymbol{\epsilon}^{\text{in}} = \boldsymbol{\epsilon}^{\text{e}} + \boldsymbol{\epsilon}^{\text{d}} + \boldsymbol{\epsilon}^{\text{p}} \tag{3.62}$$

This is the kinematic decomposition suggested by Armero and Oller [2]. Accordingly, the above embedded/smeared discontinuity models fit the framework developed in that reference. As shown in next section, an equivalent traction-based elastoplastic damage model can be developed based on an alternative kinematic decomposition. □

4. Strain localization of inelastic solids

In the methodology employed in Section 3, it is implicitly assumed *a priori* that strain localization occurs with a continuous stress field and the discontinuity (band) orientation is already known. However, on one hand, whether such strain localization can occur has not been checked yet; even though it can, the effects of stress continuity on the discontinuity orientation have not been identified, either. On the other hand, the traction-based failure criterion characterizing the discontinuity (band) is usually introduced in a heuristic or even *ad hoc* manner, and it is hard to determine the involved parameters from available experimental data.

In this section, an alternative methodology, i.e., strain localization in inelastic softening solids, is considered. As is well-known, for strain localization to occur in a softening solid and to develop eventually into a fully softened discontinuity at the final stage of the deformation process, material points inside the discontinuity (band) undergo inelastic loading while those outside it unload elastically [11, 38]. This fact imposes a kinematic constraint on the discontinuity kinematics. Application of the resulting kinematic condition to an inelastic material model, e.g., a unified elastoplastic damage model considered in this section, naturally yields all the ingredients characterizing the discontinuity (band), i.e., localized constitutive laws, traction-based failure criterion and discontinuity orientation, etc.

4.1. Kinematic constraint upon strain localization

To discuss the stress field $\boldsymbol{\sigma}$ around the discontinuity (band), let us consider the generic inelastic constitutive relations (3.3) or the rate forms (3.4). Owing to the strain continuity (2.4) at both sides of the discontinuity (band), the stresses $\boldsymbol{\sigma}_{\mathcal{S}}^{+} := \boldsymbol{\sigma}(\boldsymbol{x} \in \Omega^{+} \cap \mathcal{S}^{+})$ "ahead of" the surface $\mathcal{S}^{+}$ and $\boldsymbol{\sigma}_{\mathcal{S}}^{-} := \boldsymbol{\sigma}(\boldsymbol{x} \in \Omega^{-} \cap \mathcal{S}^{-})$ "behind" the surface $\mathcal{S}^{-}$ are equal and denoted by $\boldsymbol{\sigma}_{\mathcal{S}}^{\pm}$. Being the material outside the discontinuity (band) elastic, they are determined as

$$\boldsymbol{\sigma}_{\mathcal{S}}^{+} = \boldsymbol{\sigma}_{\mathcal{S}}^{-} = \boldsymbol{\sigma}_{\mathcal{S}}^{\pm} = \mathbb{E}^{0} : \boldsymbol{\epsilon}_{\mathcal{S}}^{\pm} \tag{4.1}$$

Comparatively, the stress $\boldsymbol{\sigma}_{\mathcal{S}} := \boldsymbol{\sigma}(\boldsymbol{x} \in \mathcal{S})$ at the discontinuity (band) is expressed as

$$\boldsymbol{\sigma}_{\mathcal{S}} = \mathbb{E}^{0} : \boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{e}} = \mathbb{E}^{0} : \left(\boldsymbol{\epsilon}_{\mathcal{S}} - \boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{in}}\right) \tag{4.2}$$

for the corresponding elastic and inelastic strains $(\boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{e}}, \boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{in}})$. Accordingly, the stress jump $[\![\boldsymbol{\sigma}]\!] := \boldsymbol{\sigma}_{\mathcal{S}} - \boldsymbol{\sigma}_{\mathcal{S}}^{\pm}$ is given by

$$[\![\boldsymbol{\sigma}]\!] = \mathbb{E}^{0} : \left([\![\boldsymbol{\epsilon}]\!] - \boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{in}}\right) = \mathbb{E}^{0} : \left[(\boldsymbol{e} \otimes \boldsymbol{n})^{\mathrm{sym}} - \boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{in}}\right] \tag{4.3}$$

where the strain jump $[\![\boldsymbol{\epsilon}]\!]$ is described by Maxwell's compatibility condition (2.5).

Once the discontinuity (band) forms, the traction continuity condition $(2.6)_2$ has to be fulfilled, i.e.,

$$[\![\boldsymbol{t}]\!] := \boldsymbol{\sigma}_{\mathcal{S}} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_{\mathcal{S}}^{\pm} \cdot \boldsymbol{n} = [\![\boldsymbol{\sigma}]\!] \cdot \boldsymbol{n} = \boldsymbol{0} \tag{4.4}$$

Note that, owing to the relation (4.1) traction continuity between $\boldsymbol{\sigma}_{\mathcal{S}}^{+} \cdot \boldsymbol{n}$ and $\boldsymbol{\sigma}_{\mathcal{S}}^{-} \cdot \boldsymbol{n}$, necessary for the balance laws [37, 57], is automatically satisfied.

As the strains $\boldsymbol{\epsilon}_{\mathcal{S}}^{\pm}$ and the resulting stresses $\boldsymbol{\sigma}_{\mathcal{S}}^{\pm}$ outside the discontinuity (band) are regular, the traction continuity condition $(2.6)_2$ requires that both the traction $\boldsymbol{t} = \boldsymbol{\sigma}_{\mathcal{S}} \cdot \boldsymbol{n}$ and the stress $\boldsymbol{\sigma}_{\mathcal{S}}$ are bounded [39]. Therefore, though the strain jump $[\![\boldsymbol{\epsilon}]\!]$ and the inelastic strain $\boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{in}}$ are both inversely proportional to the band width b (see Remark 4.4), the stress jump $[\![\boldsymbol{\sigma}]\!]$ has to be regular (bounded) and independent of b.

For the strong discontinuity with a vanishing band width $b \to 0$, stress boundedness requires *cancellation of the unbounded strain jump* [11, 38, 39], i.e.,

$$\boxed{[\![\boldsymbol{\epsilon}]\!] = \boldsymbol{\epsilon}_{\mathcal{S}}^{\mathrm{in}} = (\boldsymbol{e} \otimes \boldsymbol{n})^{\mathrm{sym}} = \frac{1}{b}(\boldsymbol{w} \otimes \boldsymbol{n})^{\mathrm{sym}}} \tag{4.5}$$

This is equivalent to *satisfaction of the stress continuity condition* (3.5). As the stresses should not depend on the band width b, even if the deformation vector $\boldsymbol{e}$ and the strain $\boldsymbol{\epsilon}$ do, the above arguments also hold for the regularized discontinuity with a finite band width $b \nrightarrow 0$. Note that the stress continuity condition (3.5) and the equivalent kinematic constraint (4.5) can also be written in rate form.

The kinematic constraint (4.5) states that, ***if strain localization can occur with a continuous stress field, difference in the strains between interior and exterior points of the discontinuity (band), i.e., the strain jump characterized by Maxwell's compatibility condition, has to be completely inelastic***. This conclusion is consistent with the methodology employed in the strong/regularized and embedded/smeared discontinuity approaches discussed in Section 3.

Remark 4.1 For a fully softened discontinuity (band) to develop at the final stage of the deformation process (i.e., the internal variable $\kappa \to \infty$), the bulk stresses have to unload completely and the transferred tractions should vanish, i.e.,

$$\lim_{\kappa \to \infty} \boldsymbol{\sigma} = \boldsymbol{0}, \qquad \lim_{\kappa \to \infty} \boldsymbol{t} = \boldsymbol{0} \tag{4.6}$$

In this case, the stress continuity condition (3.5) is also necessary for satisfaction of the above decohesion condition. Contrariwise, the traction continuity condition $(2.6)_2$ generally guarantees neither the stress continuity (3.5) nor the limit decohesion (4.6). Therefore, the stress continuity condition (3.5) is more stringent than the traction continuity condition $(2.6)_2$. □

4.2. A unified elastoplastic damage framework

To analyze strain localization in general inelastic softening solids, a unified elastoplastic damage framework [30, 67] is presented in this section. Both stress- and traction-based elastoplastic damage models can be developed within this framework.

4.2.1. Stress–strain relations

To account for both damage evolution and plastic flows, the constitutive relations are described by the following elastoplastic damage model

$$\boldsymbol{\sigma} = \mathbb{E} : \left(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\mathrm{p}}\right), \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} + \boldsymbol{\epsilon}^{\mathrm{p}} \tag{4.7}$$

where the fourth-order tensors $\mathbb{E}$ and $\mathbb{C} = \mathbb{E}^{-1}$ denote the (variable) material stiffness and compliance, respectively; $\boldsymbol{\epsilon}^{\mathrm{p}}$ represents the irreversible plastic strain tensor. Note that both the compliance $\mathbb{C}$ (or the stiffness $\mathbb{E}$) and the plastic strain $\boldsymbol{\epsilon}^{\mathrm{p}}$ are internal variables.

By time differentiation, it follows that

$$\dot{\boldsymbol{\sigma}} = \mathbb{E} : \left(\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^{\mathrm{dis}}\right), \qquad \dot{\boldsymbol{\epsilon}} = \mathbb{C} : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\epsilon}}^{\mathrm{dis}} \tag{4.8}$$

where the *dissipative strain rate* $\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}}$ is defined as

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}} := \dot{\mathbb{C}} : \boldsymbol{\sigma} + \dot{\boldsymbol{\epsilon}}^{\mathrm{p}} \tag{4.9}$$

with $\dot{\mathbb{C}} : \boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}^{\mathrm{p}}$ being its damage and plastic components, respectively; see Fig. 6. Note that the dissipative strain rate $\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}}$ does not correspond to an actual "strain". It is only defined in rate form when the involved energy dissipation mechanisms, i.e., damage evolution and plastic flows, are active.

Besides the above constitutive relations, the evolution laws for the compliance tensor $\mathbb{C}$ (or the damage one $\mathbb{C}^{\mathrm{d}}$) and strain-like internal variables $\left\{\boldsymbol{\epsilon}^{\mathrm{p}}, \kappa\right\}$ have to be postulated.

Remark 4.2 As shown in Fig. 7, the strain tensor $\boldsymbol{\epsilon}$ can also be rewritten as the kinematic decomposition (3.62), in which the elastic and inelastic strain tensors $(\boldsymbol{\epsilon}^{\mathrm{e}}, \boldsymbol{\epsilon}^{\mathrm{in}})$ are expressed as

$$\epsilon^{\mathrm{e}} = \mathbb{C}^0 : \boldsymbol{\sigma}, \qquad \boldsymbol{\epsilon}^{\mathrm{in}} = \boldsymbol{\epsilon}^{\mathrm{d}} + \boldsymbol{\epsilon}^{\mathrm{p}} = \mathbb{C}^{\mathrm{d}} : \boldsymbol{\sigma} + \boldsymbol{\epsilon}^{\mathrm{p}} \tag{4.10}$$

where the damage strain tensor $\epsilon^{\mathrm{d}} := \mathbb{C}^{\mathrm{d}} : \boldsymbol{\sigma}$ represents the recoverable inelastic strain; the fourth-order damage compliance $\mathbb{C}^{\mathrm{d}}$ is defined as $\mathbb{C}^{\mathrm{d}} := \mathbb{C} - \mathbb{C}^0$, with identical evolution law $\dot{\mathbb{C}}^{\mathrm{d}} = \dot{\mathbb{C}}$. □

4.2.2. Evolution laws

Without loss of generality, let us consider a rate-independent inelastic solid characterized by the following convex, smooth and differentiable failure criterion $\mathcal{F}(\boldsymbol{\sigma}, q) \leq 0$, where the failure function $\mathcal{F}(\boldsymbol{\sigma}, q)$ is homogeneous of degree $M \geq 1$, i.e.,

$$\mathcal{F}(\boldsymbol{\sigma}, q) = \frac{1}{M}\big(\partial_{\boldsymbol{\sigma}}\mathcal{F} : \boldsymbol{\sigma} + \partial_q \mathcal{F} \cdot q\big) = \frac{1}{M}\big(\boldsymbol{\Lambda} : \boldsymbol{\sigma} - h \cdot q\big) \leq 0 \tag{4.11}$$

for the derivatives $\boldsymbol{\Lambda} := \partial\mathcal{F}/\partial\boldsymbol{\sigma}$ and $h := -\partial\mathcal{F}/\partial q$. Here, the stress-like internal variable $q(\kappa)$ characterizes the softening behavior of the material, with κ being the conjugate strain-like one. The homogeneous failure function (4.11), frequently encountered in practice, automatically guarantees the energy dissipation inequality (**??**) for any softening law $q(\kappa)$; see Wu and Cervera [66] for more details.

Similarly to the cohesive models for strong/regularized discontinuities, the following associated evolution laws are derived from the postulate of maximum energy dissipation [56] as

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}} = \lambda\boldsymbol{\Lambda}, \qquad \dot{\kappa} = \lambda h \tag{4.12}$$

where the Lagrangian multiplier λ satisfies the Kuhn-Tucker loading/unloading conditions

$$\lambda \geq 0, \qquad \mathcal{F}(\boldsymbol{\sigma}, q) \leq 0, \qquad \lambda\mathcal{F}(\boldsymbol{\sigma}, q) = 0 \tag{4.13}$$

To differentiate the damage and plastic contributions to the dissipative strain rate $\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}}$, a material parameter $\xi \in [0, 1]$ is introduced so that [30, 44, 67]

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{p}} = \big(1-\xi\big)\,\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}} = \big(1-\xi\big)\,\lambda\boldsymbol{\Lambda} \tag{4.14a}$$

$$\dot{\mathbb{C}} : \boldsymbol{\sigma} = \xi\,\dot{\boldsymbol{\epsilon}}^{\mathrm{dis}} = \xi\,\lambda\boldsymbol{\Lambda} \tag{4.14b}$$

The cases $\xi = 0$ and $\xi = 1$ correspond to the classical plasticity model and elastic damage model [8], respectively. For the intermediate value $\xi \in (0, 1)$, both the material compliance $\mathbb{C}$ (or the damage one $\mathbb{C}^{\mathrm{d}}$) and the plastic strain $\boldsymbol{\epsilon}^{\mathrm{p}}$ are internal variables, resulting in a combined plastic-damage model.

The principle of maximum dissipation determines only the components of the compliance $\mathbb{C}$ appearing in the product $\boldsymbol{\sigma} : \dot{\mathbb{C}} : \boldsymbol{\sigma}$, leaving the remaining ones undefined. A particular evolution law for the compliance $\mathbb{C}$ (or the damage one $\mathbb{C}^{\mathrm{d}}$) satisfying Eq. (4.14b) is given by [30, 67]

$$\dot{\mathbb{C}} = \dot{\mathbb{C}}^{\mathrm{d}} = \xi\,\lambda\frac{\boldsymbol{\Lambda}\otimes\boldsymbol{\Lambda}}{\boldsymbol{\Lambda} : \boldsymbol{\sigma}} \tag{4.15}$$

as long as the condition $\boldsymbol{\Lambda} : \boldsymbol{\sigma} \neq 0$ is satisfied.

Remark 4.3 Being coaxial to the stress tensor $\boldsymbol{\sigma}$, the dissipative flow tensor $\boldsymbol{\Lambda} := \partial\mathcal{F}/\partial\boldsymbol{\sigma}$ is spectrally decomposed as [23]

$$\boldsymbol{\Lambda} = \sum_{i=1}^{3} \Lambda_i\,\boldsymbol{v}_i \otimes \boldsymbol{v}_i \tag{4.16}$$

for the principal values $\Lambda_i = \partial \mathcal{F} / \partial \sigma_i$ and the identical principal vectors $\boldsymbol{v}_i$. It can also be expanded in the local base system $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{p})$, i.e.,

$$\begin{aligned}\boldsymbol{\Lambda} &= \Lambda_{nn} \boldsymbol{n} \otimes \boldsymbol{n} + 2\Lambda_{nm} \big(\boldsymbol{n} \otimes \boldsymbol{m}\big)^{\text{sym}} + 2\Lambda_{np} \big(\boldsymbol{n} \otimes \boldsymbol{p}\big)^{\text{sym}} \\ &\quad + \Lambda_{mm} \boldsymbol{m} \otimes \boldsymbol{m} + \Lambda_{pp} \boldsymbol{p} \otimes \boldsymbol{p} + 2\Lambda_{mp} \big(\boldsymbol{m} \otimes \boldsymbol{p}\big)^{\text{sym}}\end{aligned} \tag{4.17}$$

where the tensorial components are given by

$$\Lambda_{nn} = \big(\boldsymbol{n} \otimes \boldsymbol{n}\big) : \boldsymbol{\Lambda}, \qquad \Lambda_{nm} = \big(\boldsymbol{n} \otimes \boldsymbol{m}\big)^{\text{sym}} : \boldsymbol{\Lambda}, \qquad \Lambda_{np} = \big(\boldsymbol{n} \otimes \boldsymbol{p}\big)^{\text{sym}} : \boldsymbol{\Lambda} \tag{4.18a}$$

$$\Lambda_{mm} = \big(\boldsymbol{m} \otimes \boldsymbol{m}\big) : \boldsymbol{\Lambda}, \qquad \Lambda_{pp} = \big(\boldsymbol{p} \otimes \boldsymbol{p}\big) : \boldsymbol{\Lambda}, \qquad \Lambda_{mp} = \big(\boldsymbol{m} \otimes \boldsymbol{p}\big)^{\text{sym}} : \boldsymbol{\Lambda} \tag{4.18b}$$

Recalling the discontinuity orientation introduced in Section 2.3, the above tensorial components can be expressed in terms of the principal values Λ_i and the characteristic angles $\boldsymbol{\theta}$. □

4.2.3. Rate constitutive relations

When the material is unloading, i.e., $\mathcal{F}(\boldsymbol{\sigma}, q) < 0$, it follows that $\lambda = 0$; for the loading case, it follows that $\mathcal{F}(\boldsymbol{\sigma}, q) = 0$ and $\lambda > 0$ is solved from the consistency condition $\dot{\mathcal{F}}(\boldsymbol{\sigma}, q) = 0$, i.e.,

$$\dot{\mathcal{F}}(\boldsymbol{\sigma}, q) = \boldsymbol{\Lambda} : \dot{\boldsymbol{\sigma}} - \lambda h \cdot H \cdot h = 0 \tag{4.19}$$

or, equivalently,

$$\lambda = \frac{\boldsymbol{\Lambda} : \mathbb{E} : \dot{\boldsymbol{\epsilon}}}{\boldsymbol{\Lambda} : \mathbb{E} : \boldsymbol{\Lambda} + h \cdot H \cdot h} = \frac{\boldsymbol{\Lambda} : \dot{\boldsymbol{\sigma}}}{h \cdot H \cdot h} \tag{4.20}$$

Therefore, the rate constitutive relations are given by

$$\dot{\boldsymbol{\sigma}} = \mathbb{E} : \big(\dot{\boldsymbol{\epsilon}} - \lambda \boldsymbol{\Lambda}\big) = \mathbb{E}_{\text{tan}} : \dot{\boldsymbol{\epsilon}} \tag{4.21a}$$

$$\dot{\boldsymbol{\epsilon}} = \mathbb{C} : \dot{\boldsymbol{\sigma}} + \lambda \boldsymbol{\Lambda} = \mathbb{C}_{\text{tan}} : \dot{\boldsymbol{\sigma}} \tag{4.21b}$$

where the tangent stiffness and compliance are expressed as

$$\mathbb{E}_{\text{tan}} = \mathbb{E} - \frac{\mathbb{E} : \big(\boldsymbol{\Lambda} \otimes \boldsymbol{\Lambda}\big) : \mathbb{E}}{\boldsymbol{\Lambda} : \mathbb{E} : \boldsymbol{\Lambda} + h \cdot H \cdot h} \tag{4.22a}$$

$$\mathbb{C}_{\text{tan}} = \mathbb{C} + \frac{\boldsymbol{\Lambda} \otimes \boldsymbol{\Lambda}}{h \cdot H \cdot h} \tag{4.22b}$$

for the loading state (i.e., $\lambda > 0$), both being symmetric due to the associated evolution laws assumed.

4.2.4. Fracture energy

For the above elastoplastic damage model, the external energy density supplied during the deformation process $T \in [0, \infty]$ is evaluated as [66]

$$\int_0^{\infty} \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \, \mathrm{d}T = \int_0^{\infty} \left(\frac{1}{2} \boldsymbol{\sigma} : \dot{\mathbb{C}}^{\text{d}} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^{\text{p}} \right) \mathrm{d}T \tag{4.23}$$

Recalling the evolution law $(4.12)_2$ and (4.14), it is possible to define the following specific fracture energy g_{f}

$$g_{\text{f}} = \int_0^\infty \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \, \mathrm{d}T = \Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty \lambda \boldsymbol{\Lambda} : \boldsymbol{\sigma} \, \mathrm{d}T = \Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty q(\kappa) \, \mathrm{d}\kappa = \frac{G_{\text{f}}}{b} \tag{4.24}$$

where the relation $\boldsymbol{\Lambda} : \boldsymbol{\sigma} = h \cdot q$ upon satisfaction of the failure criterion $\mathcal{F}(\boldsymbol{\sigma}, q) = 0$ has been considered; G_{f} is the fracture energy (i.e., energy dissipation per unit discontinuity area), assumed as a material property. Therefore, the softening law $q(\kappa)$ has to be normalized with respect to the discontinuity band width b, so that the energy dissipation during the whole failure process does not depend on it.

The above normalization procedure was advocated in the crack band theory [5]. It is equivalent to the cohesive (fictitious) crack model [3, 4, 17, 22]. In this latter context, Eq. (4.24) is rewritten as

$$G_{\text{f}} = b g_{\text{f}} = \Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty q(\kappa) b \, \mathrm{d}\kappa = \Big(1 - \frac{1}{2}\xi\Big) \int_0^\infty q(\tilde{\kappa}) \, \mathrm{d}\tilde{\kappa} \tag{4.25}$$

Here, the residual strength $q(\tilde{\kappa})$ is expressed in terms of a *displacement*-like internal variable $\tilde{\kappa}$ defined as

$$\tilde{\kappa} := b\kappa, \qquad \dot{\tilde{\kappa}} = \tilde{\lambda} h \qquad \Longleftrightarrow \qquad \lambda = \frac{1}{b}\tilde{\lambda}, \qquad \frac{1}{H} = \frac{1}{b}\frac{1}{\widetilde{H}} \tag{4.26}$$

where $\tilde{\lambda} \geq 0$ is an alternative Lagrangian multiplier; $H := \partial q / \partial \kappa \leq 0$ and $\widetilde{H} := \partial q / \partial \tilde{\kappa} \leq 0$ denote the strain- and displacement-driven softening moduli, respectively. The above relations (4.26) coincide with the equivalence conditions (3.47) between the cohesive laws for strong and regularized discontinuities.

As can be seen, ***both the strain-driven softening law*** $q(\kappa)$ ***and the displacement-driven one*** $q(\tilde{\kappa})$ ***are equivalent and give identical results for the same fracture energy*** G_{f}. In stress-based models the strain-driven one $q(\kappa)$ is generally preferred, with the involved parameter determined through the crack band theory (4.24). In traction-based models the displacement-driven one $q(\tilde{\kappa})$ and the cohesive crack model (4.25) are usually employed.

Remark 4.4 It is concluded from Eq. (4.26) that all the kinematic internal variables characterizing the inelastic behavior of the material, e.g., the damage compliance $\mathbb{C}^{\text{d}}$, the plastic strain tensor $\boldsymbol{\epsilon}^{\text{p}}$ and the inelastic strain tensor $\boldsymbol{\epsilon}^{\text{in}}$, etc., are inversely proportional to the discontinuity band width b. □

4.3. Strain localization analysis of the elastoplastic damage solid

Let us now consider strain localization in an inelastic solid characterized by the above elastoplastic damage model. To this end, the kinematic constraint (4.5) has to be accounted for appropriately. More specifically, upon strain localization the dissipative flow tensor characterizing the evolution laws evolves into a particular structure expressed uniquely in terms of a dissipative flow vector and the discontinuity orientation. That is, the flow tensorial components in the directions orthogonal to the discontinuity orientation have to vanish. This property allows introducing a traction-based failure criterion and developing a localized plastic-damage model for the discontinuity (band).

4.3.1. Strain localization analysis

Recalling the inelastic strain $(4.10)_2$, the kinematic condition (4.5) particularizes into

$$\boldsymbol{\epsilon}^{\text{in}} = \big(\boldsymbol{e} \otimes \boldsymbol{n}\big)^{\text{sym}} = \boldsymbol{\epsilon}^{\text{d}} + \boldsymbol{\epsilon}^{\text{p}} = \mathbb{C}^{\text{d}} : \boldsymbol{\sigma} + \boldsymbol{\epsilon}^{\text{p}} \tag{4.27}$$

so that the stress continuity condition (3.5) is fulfilled upon strain localization.

In the right hand side of the relation (4.27), the damage/plastic strains $(\boldsymbol{\epsilon}^{\mathrm{d}}, \boldsymbol{\epsilon}^{\mathrm{p}})$ correspond to the recoverable and irreversible components of the inelastic strain $\boldsymbol{\epsilon}^{\mathrm{in}}$, respectively. Therefore, the (apparent) displacement jump $\boldsymbol{w}$ and the inelastic deformation vector $\boldsymbol{e} := \boldsymbol{w}/b$ have to admit similar decompositions

$$\boldsymbol{w} = \boldsymbol{w}^{\mathrm{d}} + \boldsymbol{w}^{\mathrm{p}}, \qquad \boldsymbol{e} = \boldsymbol{e}^{\mathrm{d}} + \boldsymbol{e}^{\mathrm{p}} \tag{4.28}$$

such that

$$(\boldsymbol{e}^{\mathrm{p}} \otimes \boldsymbol{n})^{\mathrm{sym}} = \boldsymbol{\epsilon}^{\mathrm{p}} \tag{4.29a}$$

$$(\boldsymbol{e}^{\mathrm{d}} \otimes \boldsymbol{n})^{\mathrm{sym}} = \boldsymbol{\epsilon}^{\mathrm{d}} = \mathbb{C}^{\mathrm{d}} : \boldsymbol{\sigma} \tag{4.29b}$$

where the damage and plastic deformation vectors, $\boldsymbol{e}^{\mathrm{d}} := \boldsymbol{w}^{\mathrm{d}}/b$ and $\boldsymbol{e}^{\mathrm{p}} := \boldsymbol{w}^{\mathrm{d}}/b$, are defined as the recoverable and unrecoverable displacement jumps $(\boldsymbol{w}^{\mathrm{d}}, \boldsymbol{w}^{\mathrm{p}})$ normalized with respect to the band width b.

It follows from Eqs. (4.14a) and (4.29a) that

$$(\dot{\boldsymbol{e}}^{\mathrm{p}} \otimes \boldsymbol{n})^{\mathrm{sym}} = \dot{\boldsymbol{\epsilon}}^{\mathrm{p}} = (1 - \xi)\lambda \boldsymbol{\Lambda} \tag{4.30}$$

Satisfaction of the above relation implies the existence of a *dissipative flow vector* $\boldsymbol{\gamma}$ satisfying

$$\dot{\boldsymbol{e}}^{\mathrm{p}} = (1 - \xi)\lambda \boldsymbol{\gamma}, \qquad (\boldsymbol{\gamma} \otimes \boldsymbol{n})^{\mathrm{sym}} = \boldsymbol{\Lambda} \tag{4.31}$$

Furthermore, owing to the static constraint (3.6), the following identity holds

$$\boldsymbol{\Lambda} : \boldsymbol{\sigma} = \boldsymbol{\gamma} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n}) = \boldsymbol{\gamma} \cdot \boldsymbol{t} \tag{4.32}$$

between the dissipative flow tensor $\boldsymbol{\Lambda}$ and vector $\boldsymbol{\gamma}$. This relation is useful to link the stress- and traction-based failure criteria characterizing the discontinuity (band).

Pre-multiplying relation $(4.31)_2$ by the symmetric fourth-order identity tensor $\mathbb{I}$ and then by the orientation $\boldsymbol{n}$, it follows that [39]

$$\boldsymbol{\gamma} = 2\boldsymbol{n} \cdot \boldsymbol{\Lambda} - \boldsymbol{n}\Lambda_{nn} = \gamma_n\, \boldsymbol{n} + \gamma_m\, \boldsymbol{m} + \gamma_p\, \boldsymbol{p} \tag{4.33a}$$

where the local components $(\gamma_n, \gamma_m, \gamma_p)$ of the dissipative flow vector $\boldsymbol{\gamma}$ are expressed as

$$\gamma_n := \boldsymbol{\gamma} \cdot \boldsymbol{n} = \Lambda_{nn}, \qquad \gamma_m := \boldsymbol{\gamma} \cdot \boldsymbol{m} = 2\Lambda_{nm}, \qquad \gamma_p := \boldsymbol{\gamma} \cdot \boldsymbol{p} = 2\Lambda_{np} \tag{4.33b}$$

Substitution of the above dissipative flow vector $\boldsymbol{\gamma}$ into Eq. $(4.31)_2$ yields

$$\boxed{\boldsymbol{\Lambda} = (\boldsymbol{\gamma} \otimes \boldsymbol{n})^{\mathrm{sym}} = \Lambda_{nn}(\boldsymbol{n} \otimes \boldsymbol{n}) + 2\Lambda_{nm}(\boldsymbol{n} \otimes \boldsymbol{m})^{\mathrm{sym}} + 2\Lambda_{np}(\boldsymbol{n} \otimes \boldsymbol{p})^{\mathrm{sym}}} \tag{4.34}$$

Comparison with the dissipative flow tensor (4.17) imposes the following constraints

$$\Lambda_{mm}(\boldsymbol{\theta}^{\mathrm{cr}}) = 0, \qquad \Lambda_{pp}(\boldsymbol{\theta}^{\mathrm{cr}}) = 0, \qquad \Lambda_{mp}(\boldsymbol{\theta}^{\mathrm{cr}}) = 0 \tag{4.35}$$

where $\boldsymbol{\theta}^{\mathrm{cr}}$ denote the critical angles of the discontinuity (band).

It is concluded from Eq. (4.35) that, upon strain localization the failure criterion $\mathcal{F}(\boldsymbol{\sigma}, q) \leq 0$ does not depend on the stress components σ_{mm}, σ_{pp} and σ_{mp}, but is only a function of the tractions $\boldsymbol{t} := \boldsymbol{\sigma} \cdot \boldsymbol{n} = \{\sigma_{nn}, \sigma_{nm}, \sigma_{np}\}^{\mathrm{T}}$. Therefore, as long as the critical angles $\boldsymbol{\theta}^{\mathrm{cr}}$ satisfying the kinematic constraints (4.35) exist, it is always possible to introduce an appropriate traction-based failure criterion to characterize the discontinuity (band); see Section 4.4.

4.3.2. *Localized constitutive model for the discontinuity (band)*

Owing to the particular structure (4.34) of the dissipative flow tensor $\boldsymbol{\Lambda}$ upon strain localization, the evolution laws (4.12) and (4.14) are naturally projected to the orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\mathrm{cr}})$. This property allows developing a localized plastic-damage model for the discontinuity (band).

With the dissipative flow tensor (4.34), the damage evolution law (4.14b) can be rewritten as

$$\dot{\mathbb{C}} = \dot{\mathbb{C}}^{\mathrm{d}} = \big(\dot{\boldsymbol{C}}^{\mathrm{d}} \,\overline{\underline{\otimes}}\, \boldsymbol{N}\big)^{\mathrm{sym}} \qquad \Longrightarrow \qquad \mathbb{C}^{\mathrm{d}} = \big(\boldsymbol{C}^{\mathrm{d}} \,\overline{\underline{\otimes}}\, \boldsymbol{N}\big)^{\mathrm{sym}} \tag{4.36}$$

where $\boldsymbol{N} := \boldsymbol{n} \otimes \boldsymbol{n}$ is a second-order geometric tensor; $\boldsymbol{C}^{\mathrm{d}}$ denotes the second-order compliance tensor of the discontinuity (band), with the following evolution law

$$\dot{\boldsymbol{C}}^{\mathrm{d}} = \xi\, \lambda \frac{\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}}{\boldsymbol{\gamma} \cdot \boldsymbol{t}} \tag{4.37}$$

In other words, upon strain localization the damage behavior of the material is sufficiently characterized by a second-order internal variable $\boldsymbol{C}^{\mathrm{d}}$ rather than the original fourth-order one $\mathbb{C}^{\mathrm{d}}$; see Remark 4.5 for details.

Accordingly, the damage strain tensor (4.29b) becomes

$$\boldsymbol{\epsilon}^{\mathrm{d}} = \big(\boldsymbol{e}^{\mathrm{d}} \otimes \boldsymbol{n}\big)^{\mathrm{sym}} = \mathbb{C}^{\mathrm{d}} : \boldsymbol{\sigma} = \Big[\big(\boldsymbol{C}^{\mathrm{d}} \cdot \boldsymbol{t}\big) \otimes \boldsymbol{n}\Big]^{\mathrm{sym}} \tag{4.38}$$

where the static constraint (3.6) has been considered. That is, the discontinuity (band) can be described by the following localized plastic-damage constitutive relations

$$\boldsymbol{e}^{\mathrm{d}} = \boldsymbol{e} - \boldsymbol{e}^{\mathrm{p}} = \boldsymbol{C}^{\mathrm{d}} \cdot \boldsymbol{t}, \qquad \boldsymbol{t} = \boldsymbol{E}^{\mathrm{d}} \cdot \boldsymbol{e}^{\mathrm{d}} = \boldsymbol{E}^{\mathrm{d}} \cdot \big(\boldsymbol{e} - \boldsymbol{e}^{\mathrm{p}}\big) \tag{4.39}$$

for the second-order stiffness tensor $\boldsymbol{E}^{\mathrm{d}} := \big(\boldsymbol{C}^{\mathrm{d}}\big)^{-1}$ and the localized plastic evolution law (4.31).

By time differentiation, it follows that

$$\dot{\boldsymbol{t}} = \boldsymbol{E}^{\mathrm{d}} \cdot \big(\dot{\boldsymbol{e}} - \dot{\boldsymbol{e}}^{\mathrm{dis}}\big), \qquad \dot{\boldsymbol{e}} = \boldsymbol{C}^{\mathrm{d}} \cdot \dot{\boldsymbol{t}} + \dot{\boldsymbol{e}}^{\mathrm{dis}} \tag{4.40}$$

where the *dissipative deformation vector rate* $\dot{\boldsymbol{e}}^{\mathrm{dis}}$ is expressed as

$$\dot{\boldsymbol{e}}^{\mathrm{dis}} := \dot{\boldsymbol{C}}^{\mathrm{d}} \cdot \boldsymbol{t} + \dot{\boldsymbol{e}}^{\mathrm{p}} = \lambda \boldsymbol{\gamma} \qquad \Longrightarrow \qquad \big(\dot{\boldsymbol{e}}^{\mathrm{dis}} \otimes \boldsymbol{n}\big)^{\mathrm{sym}} = \dot{\boldsymbol{\epsilon}}^{\mathrm{dis}} = \lambda \boldsymbol{\Lambda} \tag{4.41}$$

with $\dot{\boldsymbol{C}}^{\mathrm{d}} \cdot \boldsymbol{t}$ and $\dot{\boldsymbol{e}}^{\mathrm{p}}$ being its damage and plastic components, respectively; see Fig. 5.

For an active discontinuity, it follows that $\mathcal{F}(\boldsymbol{\sigma}, q) = 0$. The corresponding consistency condition (4.19) becomes

$$\dot{\mathcal{F}} = (\boldsymbol{\gamma} \otimes \boldsymbol{n})^{\text{sym}} : \dot{\boldsymbol{\sigma}} - h \cdot H \cdot h = \boldsymbol{\gamma} \cdot \dot{\boldsymbol{t}} - h \cdot H \cdot h = 0 \tag{4.42}$$

from which the multiplier $\lambda > 0$ is solved in terms of the inelastic deformation rate $\dot{\boldsymbol{e}}$ instead of the strain rate $\dot{\boldsymbol{\epsilon}}$ given in Eq. (4.20)

$$\lambda = \frac{\boldsymbol{\gamma} \cdot \boldsymbol{E}^{\text{d}} \cdot \dot{\boldsymbol{e}}}{\boldsymbol{\gamma} \cdot \boldsymbol{E}^{\text{d}} \cdot \boldsymbol{\gamma} + h \cdot H \cdot h} = \frac{\boldsymbol{\gamma} \cdot \dot{\boldsymbol{t}}}{h \cdot H \cdot h} \tag{4.43}$$

Combination of Eqs. (4.40), (4.41) and (4.43) yields the following rate constitutive relations

$$\dot{\boldsymbol{t}} = \boldsymbol{E}^{\text{d}} \cdot (\dot{\boldsymbol{e}} - \lambda \boldsymbol{\gamma}) = \boldsymbol{E}^{\text{d}}_{\text{tan}} \cdot \dot{\boldsymbol{e}} \tag{4.44a}$$

$$\dot{\boldsymbol{e}} = \boldsymbol{C}^{\text{d}} \cdot \dot{\boldsymbol{t}} + \lambda \boldsymbol{\gamma} = \boldsymbol{C}^{\text{d}}_{\text{tan}} \cdot \dot{\boldsymbol{t}} \tag{4.44b}$$

where the tangent stiffness and compliance are expressed as

$$\boldsymbol{E}^{\text{d}}_{\text{tan}} = \boldsymbol{E}^{\text{d}} - \frac{\boldsymbol{E}^{\text{d}} \cdot (\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}) \cdot \boldsymbol{E}^{\text{d}}}{\boldsymbol{\gamma} \cdot \boldsymbol{E}^{\text{d}} \cdot \boldsymbol{\gamma} + h \cdot H \cdot h} \tag{4.45a}$$

$$\boldsymbol{C}^{\text{d}}_{\text{tan}} = \boldsymbol{C}^{\text{d}} + \frac{\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}}{h \cdot H \cdot h} \tag{4.45b}$$

for an active discontinuity (band).

The above localized constitutive laws are expressed in terms of the traction $\boldsymbol{t}$ and the inelastic deformation vector $\boldsymbol{e}$. Recalling the relations (4.26), equivalent localized constitutive laws in terms of the traction $\boldsymbol{t}$ and the displacement jump $\boldsymbol{w}$ can also be developed. The details are omitted here.

As can be seen, ***as long as strain localization occurs with a continuous stress field, the discontinuity (band) can be characterized by a localized model which is formally identical but not necessarily equivalent to that in the strong/regularized discontinuity approaches***.

Remark 4.5 With the damage compliance tensor $(4.36)_2$, the material compliance tensor $\mathbb{C}$ is given by

$$\mathbb{C} = \mathbb{C}^0 + \mathbb{C}^{\text{d}} = \mathbb{C}^0 + \left(\boldsymbol{C}^{\text{d}} \,\overline{\underline{\otimes}}\, \boldsymbol{N}\right)^{\text{sym}} \tag{4.46a}$$

Inversely, the material stiffness tensor $\mathbb{E}$ is obtained from the Sherman-Morrison-Woodburg formula [18]

$$\mathbb{E} = \mathbb{C}^{-1} = \mathbb{E}^0 - \mathbb{E}^0 : \left[\left(\boldsymbol{E}^{\text{d}} + \boldsymbol{n} \cdot \mathbb{E}^0 \cdot \boldsymbol{n}\right)^{-1} \overline{\underline{\otimes}}\, \boldsymbol{N}\right]^{\text{sym}} : \mathbb{E}^0 \tag{4.46b}$$

Note the similarity between above overall material stiffness/compliance tensors and the tangent ones, i.e., Eqs. (3.56) and (3.59), obtained in the embedded/smeared discontinuity models in Section 3.4. □

4.4. Failure criterion

Dependent on the strategies dealing with the discontinuity orientation and the associated traction-based failure criterion, two dual approaches, not necessarily equivalent, can be developed.

4.4.1. Traction-based elastoplastic damage model

Providing the characteristic angles $\boldsymbol{\theta}^{\text{cr}}$ of the discontinuity (band) are known *a priori*, say, $\boldsymbol{\theta}^{\text{cr}} = \widehat{\boldsymbol{\theta}}^{\text{cr}}$, an explicit traction-based failure function can be introduced as in Section 3.2, i.e.,

$$\mathcal{F}(\boldsymbol{\sigma}, q) = \widehat{f}(\boldsymbol{\sigma} \cdot \boldsymbol{n}, q) = \widehat{f}(\boldsymbol{t}, q) \leq 0 \tag{4.47}$$

Accordingly, the dissipative flow tensor $\boldsymbol{\Lambda} := \partial \mathcal{F} / \partial \boldsymbol{\sigma}$ is evaluated as

$$\boldsymbol{\Lambda} = \frac{\partial \widehat{f}}{\partial \boldsymbol{\sigma}} = \frac{\partial \widehat{f}}{\partial \boldsymbol{t}} \cdot \frac{\partial \boldsymbol{t}}{\partial \boldsymbol{\sigma}} = \widehat{\boldsymbol{\gamma}} \cdot \boldsymbol{\mathcal{N}} = \left(\widehat{\boldsymbol{\gamma}} \otimes \boldsymbol{n}\right)^{\text{sym}} \tag{4.48a}$$

or in the form (4.34)

$$\boxed{\boldsymbol{\Lambda} := \left(\boldsymbol{\gamma} \otimes \boldsymbol{n}\right)^{\text{sym}}, \qquad \boldsymbol{\gamma} = \widehat{\boldsymbol{\gamma}} := \frac{\partial \widehat{f}}{\partial \boldsymbol{t}}} \tag{4.48b}$$

where the Cartesian components of the third-order tensor $\boldsymbol{\mathcal{N}} = \partial \boldsymbol{t} / \partial \boldsymbol{\sigma}$ is expressed as $\mathcal{N}_{ijk} = \frac{1}{2}\left(\delta_{ij} n_k + \delta_{ik} n_j\right)$ for the Kronecker-delta δ_{ij} and the components n_k of the normal vector $\boldsymbol{n}$.

For the dissipative flow tensor (4.48), the constitutive laws presented in Section 4.2 particularizes to the traction-based anisotropic elastoplastic damage model suggested in Wu and Xu [67]. In this localized model, the kinematic constraint (4.5) holds for any arbitrary orientation $\boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}})$ so that the stress continuity condition (3.5) is always guaranteed during the whole failure process, i.e.,

$$\boldsymbol{\Lambda} := \left(\widehat{\boldsymbol{\gamma}} \otimes \boldsymbol{n}\right)^{\text{sym}} \qquad \Longrightarrow \qquad [\![\boldsymbol{\sigma}]\!] = \boldsymbol{0} \tag{4.49}$$

In other words, it is implicitly assumed that strain localization can occur with a continuous stress field. This approach coincides with the first methodology upon which the strong/regularized and embedded/smeared discontinuity approaches are based. Similarly, the discontinuity orientation $\boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}})$ cannot be determined uniquely based on the given traction-based failure criterion $\widehat{f}(\boldsymbol{t}, q) \leq 0$, unless extra auxiliary conditions are introduced, e.g., the Mohr's maximization postulate as in Section 3.2.

With the dissipative flow tensor (4.48), it follows from the Sherman-Morrison-Woodburg formula [18] that the tangent stiffness tensors $\mathbb{E}^{\text{tan}}$ obtained in Eqs. (3.56) and (4.22a) are identical. Similarly, calling for the material compliance (4.46a), it can be also concluded that the tangent compliance tensors $\mathbb{C}^{\text{tan}}$ given in Eqs. (3.59) and (4.22b) coincide. Therefore, ***though they are developed based on different methodologies, the embedded/smeared discontinuity models are equivalent to the elastoplastic damage model, providing an identical traction-based failure criterion is employed in both methods***.

Remark 4.6 Owning to the relation (4.32), the given failure criterion $\widehat{f}(\boldsymbol{t}, q) \leq 0$ can be rewritten as

$$F(\boldsymbol{\sigma}, q) := \widehat{f}(\boldsymbol{t}, q) = \frac{1}{M}\left(\widehat{\boldsymbol{\gamma}} \cdot \boldsymbol{t} - \widehat{h} \cdot q\right) = \frac{1}{M}\left(\boldsymbol{\Lambda} : \boldsymbol{\sigma} - \widehat{h} \cdot q\right) \leq 0 \tag{4.50}$$

That is, once the discontinuity orientation $\boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}})$ is known, the given traction-based failure criterion $\widehat{f}(\boldsymbol{t}, q) \leq 0$ can be equivalently expressed as a stress-based counterpart $F(\boldsymbol{\sigma}, q) \leq 0$. □

4.4.2. Stress-based projected discontinuity model

Alternatively, let us consider a failure criterion $\mathcal{F}(\boldsymbol{\sigma}, q) \leq 0$ expressed in terms of stress invariants and is then projected to the *yet-unknown* discontinuity orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$. Without loss of generality, the following *stress-based* failure function is introduced

$$\mathcal{F}(\boldsymbol{\sigma}, q) := \widehat{F}(\mathcal{I}, q) \leq 0 \tag{4.51}$$

so that the dissipative flow tensor $\boldsymbol{\Lambda}$ is given by

$$\boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}} := \frac{\partial \widehat{F}}{\partial \boldsymbol{\sigma}} = \frac{\partial \widehat{F}}{\partial \sigma_1} \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + \frac{\partial \widehat{F}}{\partial I_1} \boldsymbol{I} + \frac{\partial \widehat{F}}{\partial J_2} \boldsymbol{s} + \cdots \tag{4.52}$$

where $\mathcal{I} := \{\sigma_1, I_1, J_2, \cdots\}$ collects the stress invariants, e.g., the major principal stress σ_1, the first invariant $I_1 := \text{tr}(\boldsymbol{\sigma})$ of the stress tensor $\boldsymbol{\sigma}$ and the second invariant $J_2 := \frac{1}{2}\boldsymbol{s} : \boldsymbol{s}$ of the deviatoric stress tensor $\boldsymbol{s} := \boldsymbol{\sigma} - \frac{1}{3}\text{tr}(\boldsymbol{\sigma})\boldsymbol{I}$, etc.

Accordingly, the kinematic constraint (4.34) becomes

$$\boxed{(\boldsymbol{\gamma} \otimes \boldsymbol{n})^{\text{sym}} = \boldsymbol{\Lambda}, \qquad \boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}} := \frac{\partial \widehat{F}}{\partial \boldsymbol{\sigma}}} \tag{4.53}$$

That is, *if and only if* the discontinuity orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$ and the associated dissipative flow vector $\boldsymbol{\gamma}$ satisfy the kinematic constraint (4.53) for the given dissipative flow tensor $\widehat{\boldsymbol{\Lambda}}$, i.e.,

$$[\![\boldsymbol{\sigma}]\!] = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \widehat{\boldsymbol{\Lambda}} = (\boldsymbol{\gamma} \otimes \boldsymbol{n})^{\text{sym}} \tag{4.54}$$

can strain localization occur with a continuous stress field, and vice versa.

In this case, the discontinuity orientation $\boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}})$ cannot be assumed arbitrarily as in the first approach. But rather, it has to be determined consistently from the constraints (4.35) together with the relations (4.18b) for the given stress-based dissipative flow tensor $\boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}}$. On one hand, as the resulting set of equations are nonlinear, the solution may not exist at all. That is, strain localization may not occur with a continuous stress field. On the other hand, providing the solution exists, it depends only on the given failure criterion and the stress state, but not on the elastic properties (i.e., Poisson's ratio).

Once the discontinuity orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$ is so determined, the associated dissipative flow vector $\boldsymbol{\gamma}$ can be obtained from Eqs. (4.33a) and (4.33b). Furthermore, as the relation (4.32) holds for the given dissipative flow tensor $\boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}}$, the projected traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$ can be determined as

$$f(\boldsymbol{t}, q) := \widehat{F}(\boldsymbol{\sigma}, q) = \frac{1}{M}(\widehat{\boldsymbol{\Lambda}} : \boldsymbol{\sigma} - \widehat{h} \cdot q) = \frac{1}{M}(\boldsymbol{\gamma} \cdot \boldsymbol{t} - \widehat{h} \cdot q) \leq 0 \tag{4.55}$$

Note that the above traction-based failure function $f(\boldsymbol{t}, q) \leq 0$, projected from the given stress-based one $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$, is not necessarily identical to the counterpart $\widehat{f}(\boldsymbol{t}, q) \leq 0$ explicitly assumed *a priori*.

In summary, ***providing the kinematic constant resulting from the stress continuity condition is fulfilled, consistent localized constitutive relations for the discontinuity (band) naturally emerge from the strain localization analysis of stress-based models with regularized softening***.

5. Comparisons and discussion

Though different methodologies are employed, both the traction-based embedded/smeared discontinuity models (or the equivalent elastoplastic damage model) and the stress-based projected discontinuity model can be developed within the unified framework presented in in Section 4. It is worthwhile to investigate the *bi-directional* connections between these two families of models, in particular, the conditions upon which they are equivalent.

5.1. Evolution laws

In both families of models, the inelastic behavior, completely localized within the discontinuity (band), is characterized by a dissipative flow tensor $\boldsymbol{\Lambda}$, expressed as the unified form (4.34) in terms of the dissipative flow vector $\boldsymbol{\gamma}$ and the orientation $\boldsymbol{n}$; see Eqs. (4.48) and (4.53), respectively. However, the physical interpretations are different.

For the traction-based embedded/smeared discontinuity models, the inelastic flow vector is given as $\boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}} := \partial \hat{f}/\partial \boldsymbol{t}$. In this case, the discontinuity orientation $\boldsymbol{n}(\hat{\boldsymbol{\theta}}^{\text{cr}})$ can be assumed arbitrarily. That is, the relation (4.48) imposes no kinematic constraint. But rather, it acts only as a definition, from which the dissipative flow tensor $\boldsymbol{\Lambda}$ is determined straightforwardly in such a way that the stress continuity (3.5) is always satisfied *a priori* upon strain localization.

Comparatively, in the stress-based projected discontinuity model both the dissipative flow vector $\boldsymbol{\gamma}$ and the orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$ are unknown and even might not exist at all. In this case, the relation (4.53) imposes necessary and sufficient kinematic constraint on the occurrence of strain localization with a continuous stress field. Accordingly, the discontinuity orientation $\boldsymbol{n}(\hat{\boldsymbol{\theta}}^{\text{cr}})$ can only be determined from the kinematic constraints (4.35) for the given dissipative flow tensor $\boldsymbol{\Lambda} = \hat{\boldsymbol{\Lambda}} := \partial \hat{f}/\partial \boldsymbol{\sigma}$. If such discontinuity orientation exists, the dissipative flow tensor $\boldsymbol{\gamma}$ is given from the relations (4.33). Otherwise, strain localization cannot occur with stress continuity.

In summary, ***even if the dissipative flow vectors coincide in both families of models, the resulting evolution laws, characterized by the dissipative flow tensor*** (4.34)***, are not necessarily identical, since the discontinuity orientations might be different***. An interesting question then naturally arises: whether and when would the discontinuity orientations in both families of models coincide?

5.2. Orientation of the discontinuity (band)

As mentioned before, for the embedded/smeared discontinuity models the orientation $\boldsymbol{n}(\hat{\boldsymbol{\theta}}^{\text{cr}})$ cannot be determined uniquely from the given traction-based failure criterion $\hat{f}(\boldsymbol{t}, q) \leq 0$. After recalling the relations (4.33b), the stationary condition (3.12) of the Mohr's maximization postulate can be rewritten as

$$\begin{aligned}
\frac{\partial \hat{f}}{\partial \boldsymbol{\theta}}\bigg|_{\hat{\boldsymbol{\theta}}^{\text{cr}}} &= \left(\Lambda_{nn}\frac{\partial \sigma_{nn}}{\partial \boldsymbol{\theta}} + 2\Lambda_{nm}\frac{\partial \sigma_{nm}}{\partial \boldsymbol{\theta}} + 2\Lambda_{np}\frac{\partial \sigma_{np}}{\partial \boldsymbol{\theta}}\right)_{\hat{\boldsymbol{\theta}}^{\text{cr}}} \\
&= -\left(\Lambda_{mm}\frac{\partial \sigma_{mm}}{\partial \boldsymbol{\theta}} + 2\Lambda_{mp}\frac{\partial \sigma_{mp}}{\partial \boldsymbol{\theta}} + \Lambda_{pp}\frac{\partial \sigma_{pp}}{\partial \boldsymbol{\theta}}\right)_{\hat{\boldsymbol{\theta}}^{\text{cr}}} = \boldsymbol{0}
\end{aligned} \tag{5.1}$$

Note that the identity $\boldsymbol{\Lambda} : (\partial \boldsymbol{\sigma}/\partial \boldsymbol{\theta}) = 0$, resulting from the coaxial property between the tensors $\boldsymbol{\Lambda} := \partial \hat{f}/\partial \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$, has been considered. As the failure function $\hat{f}(\boldsymbol{t}, q) \leq 0$ depends only on the tractions $\boldsymbol{t} := \{\sigma_{nn}, \sigma_{nm}, \sigma_{np}\}^{\text{T}}$, the condition (5.1) is fulfilled for arbitrary values of the remaining stress components $(\sigma_{mm}, \sigma_{mp}, \sigma_{pp})$. This fact yields

$$\Lambda_{mm}(\widehat{\boldsymbol{\theta}}^{\text{cr}}) = 0, \qquad \Lambda_{pp}(\widehat{\boldsymbol{\theta}}^{\text{cr}}) = 0, \qquad \Lambda_{mp}(\widehat{\boldsymbol{\theta}}^{\text{cr}}) = 0 \tag{5.2}$$

These relations correspond exactly to the kinematic constraints (4.35). Note that the maximization condition (3.13) should be verified for the discontinuity angles $\widehat{\boldsymbol{\theta}}^{\text{cr}}$ determined from Eqs. (5.2).

Inversely, in the projected discontinuity model with a given stress-based failure criterion, strain localization with a continuous stress field can occur, if and only if the critical angles $\boldsymbol{\theta}^{\text{cr}}$ satisfying the kinematic constraints (4.35) exist. Upon this condition, a traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$ can always be obtained by projecting the given stress-based one $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$. Accordingly, for the discontinuity (band) to form along the orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$, the projected traction-based failure criterion $f(\boldsymbol{t}, q) = 0$ is activated. Meanwhile, no discontinuity (band) forms at any other orientation $\boldsymbol{n}(\boldsymbol{\theta}) \neq \boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$. The above facts transform into

$$f\big[\boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{\theta}), q\big] \leq f\big[\boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}}), q\big] = 0 \tag{5.3}$$

That is, the tractions $\boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$ maximize the failure function $f(\boldsymbol{t}, q) = f[\boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{\theta}), q]$. Therefore, the discontinuity angles $\boldsymbol{\theta}^{\text{cr}}$ satisfying Eqs. (4.35) coincide with those determined from Mohr's maximization postulate.

Remark 5.1 In embedded/smeared discontinuity models, it is assumed *a priori* that the strain localization with a continuous stress field can always occur. Therefore, if the solution to Eqs. (5.2) does not exist, the discontinuity angles $\boldsymbol{\theta}^{\text{cr}}$ for the traction-based localized models should be determined from another set of solution to Eqs. (5.1)

$$\frac{\partial \sigma_{mm}}{\partial \boldsymbol{\theta}}\bigg|_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} = 0, \qquad \frac{\partial \sigma_{mp}}{\partial \boldsymbol{\theta}}\bigg|_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} = 0, \qquad \frac{\partial \sigma_{pp}}{\partial \boldsymbol{\theta}}\bigg|_{\widehat{\boldsymbol{\theta}}^{\text{cr}}} = 0 \tag{5.4}$$

Contrariwise, for the stress-based projected discontinuity model if the solution to the kinematic constraints (4.35) does not exist, the strain localization with a continuous stress field cannot occur. In this situation, the given stress-based failure criterion can be modified based on the solution to Eq. (5.4) in such a way that both families of models are completely equivalent; see Wu and Cervera [66] for the details. □

5.3. *Equivalence conditions between traction- and stress-based localized models*

With the above arguments, it can be concluded that:

- An embedded/smeared discontinuity model is equivalent to the projected discontinuity model with a given stress-based failure criterion $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$, providing the traction-based failure criterion in the former is adopted as $\widehat{f}(\boldsymbol{t}, q) = f(\boldsymbol{t}, q) \leq 0$ and Mohr's maximization postulate is used to determine the discontinuity orientation. Here, the traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$ is projected from the given stress-based counterpart $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$ to the orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$ which satisfies the kinematic constraint (4.53).

- A projected discontinuity model is equivalent to the embedded/smeared discontinuity model with a traction-based failure criterion $\widehat{f}(\boldsymbol{t}, q) \leq 0$ and the discontinuity orientation $\boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}})$ determined from Mohr's maximization postulate, if the stress-based failure criterion in the former is adopted as $\widehat{F}(\boldsymbol{\sigma}, q) = \widehat{f}\big[\boldsymbol{\sigma} \cdot \boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}}), q\big] \leq 0$.

In this case, the discontinuity orientation $\boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}})$ and the associated dissipative flow vector $\boldsymbol{\gamma}$ satisfying the kinematic constraint (4.53) coincide with the given ones $\boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}})$ and $\widehat{\boldsymbol{\gamma}} := \partial \widehat{f} / \partial \boldsymbol{t}$.

Upon the above conditions, the inelastic evolution laws coincide for both families of models, i.e.,

$$\boldsymbol{\gamma} = \widehat{\boldsymbol{\gamma}}, \qquad \boldsymbol{n}(\boldsymbol{\theta}^{\text{cr}}) = \boldsymbol{n}(\widehat{\boldsymbol{\theta}}^{\text{cr}}) \qquad \Longleftrightarrow \qquad \boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}} \tag{5.5}$$

Providing identical softening laws $q(\kappa)$ or $q(\bar{\kappa})$ are used, the traction-based embedded/smeared discontinuity models (or the equivalent elastoplastic damage model) and the stress-based projected discontinuity model are equivalent to each other.

6. Plane stress examples

In this section the general 3D results presented in previous sections are particularized to the plane stress condition ($\sigma_3 = 0$). The plane strain results are discussed in Wu and Cervera [66].

6.1. Preliminaries

As shown in Figure 8, the unit normal vector $\boldsymbol{n}$ and tangential vector $\boldsymbol{m}$ of the discontinuity $\mathcal{S}$ in a 2D solid $\Omega \subset \mathbb{R}^2$ are on the plane of interest. Let $\theta \in [-\pi/2, \pi/2]$ denote the angle (anti-clockwise) between the normal vector $\boldsymbol{n}$ and major principal vector $\boldsymbol{v}_1$ of the stress $\boldsymbol{\sigma}$. Accordingly, the characteristic angles $\boldsymbol{\theta} := \{\theta_1, \theta_2, \vartheta_1, \vartheta_2\}^{\text{T}}$ are characterized by the azimuth angles $\theta_1 = \theta$ and $\theta_2 = \theta - \pi/2$ and the polar angles $\vartheta_1 = \vartheta_2 = \pi/2$. It follows that $\boldsymbol{n} = \{\cos\theta, \sin\theta, 0\}^{\text{T}}$, $\boldsymbol{m} = \{\sin\theta, -\cos\theta, 0\}^{\text{T}}$ and the other tangential vector $\boldsymbol{p} = \{0, 0, -1\}^{\text{T}}$.

The in-plane stress components $(\sigma_{nn}, \sigma_{mm}, \sigma_{nm})$ are given from Eqs. (3.11)

$$\sigma_{nn} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos(2\theta) \tag{6.1a}$$

$$\sigma_{mm} = \sigma_1 \sin^2\theta + \sigma_2 \cos^2\theta = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}\cos(2\theta) \tag{6.1b}$$

$$\sigma_{nm} = (\sigma_1 - \sigma_2)\sin\theta\cos\theta = \frac{\sigma_1 - \sigma_2}{2}\sin(2\theta) \tag{6.1c}$$

with σ_1 and σ_2 being the in-plane principal stresses, ordered as $\sigma_1 \geq \sigma_2$.

As the dissipative flow tensor $\boldsymbol{\Lambda}$ is coaxial to the stress $\boldsymbol{\sigma}$, its in-plane components $(\Lambda_{nn}, \Lambda_{mm}, \Lambda_{nm})$ is similarly given from Eq. (4.18)

$$\Lambda_{nn} = \Lambda_1 \cos^2\theta + \Lambda_2 \sin^2\theta = \frac{\Lambda_1 + \Lambda_2}{2} + \frac{\Lambda_1 - \Lambda_2}{2}\cos(2\theta) \tag{6.2a}$$

$$\Lambda_{mm} = \Lambda_1 \sin^2\theta + \Lambda_2 \cos^2\theta = \frac{\Lambda_1 + \Lambda_2}{2} - \frac{\Lambda_1 - \Lambda_2}{2}\cos(2\theta) \tag{6.2b}$$

$$\Lambda_{nm} = (\Lambda_1 - \Lambda_2)\cos\theta\sin\theta = \frac{\Lambda_1 - \Lambda_2}{2}\sin(2\theta) \tag{6.2c}$$

for the in-plane principal values Λ_1 and Λ_2. The relation $\Lambda_1 \geq \Lambda_2$ is assumed so that the identity $\text{sign}(\Lambda_{nm}) = \text{sign}(\sigma_{nm})$ holds, with $\text{sign}(\cdot)$ being the sign function.

Note that in the plane stress condition all the out-of-plane components of tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\Lambda}$ vanish, i.e., $\sigma_{pp} = \sigma_{np} = \sigma_{mp} = 0$ and $\Lambda_{pp} = \Lambda_{np} = \Lambda_{mp} = 0$.

6.2. *Explicit traction-based failure criterion*

In the case of plane stress, the stationarity condition (3.12) for a given traction-based failure criterion $\widehat{f}(\boldsymbol{t}, q) \leq 0$ simplifies as

$$\left.\frac{\partial \widehat{f}}{\partial \theta}\right|_{\widehat{\theta}^{\mathrm{cr}}} = -(\sigma_1 - \sigma_2)\Big[\widehat{\gamma}_n \sin\left(2\widehat{\theta}^{\mathrm{cr}}\right) - \widehat{\gamma}_m \cos\left(2\widehat{\theta}^{\mathrm{cr}}\right)\Big] = 0 \tag{6.3}$$

The discontinuity angle $\widehat{\theta}^{\mathrm{cr}}$ is then determined from

$$\tan\left(2\widehat{\theta}^{\mathrm{cr}}\right) = \frac{\widehat{\gamma}_m}{\widehat{\gamma}_n} \qquad \text{and} \qquad \left.\frac{\partial^2 \widehat{f}}{\partial \theta^2}\right|_{\widehat{\theta}^{\mathrm{cr}}} < 0 \tag{6.4}$$

In general, two values can be obtained.

Alternatively, the discontinuity angle $\widehat{\theta}^{\mathrm{cr}}$ can also be expressed in terms of the dissipative flow tensor $\boldsymbol{\Lambda}$. It follows from the relations (4.33b) and (6.2) that

$$\widehat{\gamma}_n = \Lambda_{nn} = \frac{\Lambda_1 + \Lambda_2}{2} + \frac{\Lambda_1 - \Lambda_2}{2}\cos(2\theta) \tag{6.5a}$$

$$\widehat{\gamma}_m = 2\Lambda_{nm} = \left(\Lambda_1 - \Lambda_2\right)\sin(2\theta) \tag{6.5b}$$

The stationarity conditions (6.3) then yields

$$\cos(2\widehat{\theta}^{\mathrm{cr}}) = \frac{\Lambda_1 + \Lambda_2}{\Lambda_1 - \Lambda_2}, \qquad \sin(2\widehat{\theta}^{\mathrm{cr}}) = 2\mathrm{sign}(\sigma_{nm})\frac{\sqrt{-\Lambda_1\Lambda_2}}{\Lambda_1 - \Lambda_2} \tag{6.6}$$

if the conditions $\Lambda_1 \geq 0$ and $\Lambda_2 \leq 0$ are satisfied. For the cases $\Lambda_2 > 0$ or $\Lambda_1 < 0$, it follows that $\sin(2\widehat{\theta}^{\mathrm{cr}}) = 0$, i.e., $\widehat{\theta}^{\mathrm{cr}} = 0$ if $\Lambda_1 > \Lambda_2 > 0$ and $\widehat{\theta}^{\mathrm{cr}} = \pi/2$ if $\Lambda_2 < \Lambda_1 < 0$.

For the discontinuity angle $\widehat{\theta}^{\mathrm{cr}}$ determined from Eq. (6.6), it follows from Eqs. (6.1) and (6.5) that the identity (4.32) is recovered for $\boldsymbol{\gamma} = \widehat{\boldsymbol{\gamma}}$, i.e.,

$$\widehat{\boldsymbol{\gamma}} \cdot \boldsymbol{t} = \widehat{\gamma}_n t_n + \widehat{\gamma}_m t_m = \Lambda_1\sigma_1 + \Lambda_2\sigma_2 = \boldsymbol{\Lambda} : \boldsymbol{\sigma} \tag{6.7}$$

so that the equivalent stress-based failure criterion (4.50) becomes

$$F(\boldsymbol{\sigma}, q) = \frac{1}{M}\Big[\left(\Lambda_1\sigma_1 + \Lambda_2\sigma_2\right) - \widehat{h} \cdot q\Big] \leq 0 \tag{6.8}$$

where the relation $\Lambda_3\sigma_3 = 0$, fulfilled in the case of plane stress ($\sigma_3 = 0$), has been considered.

6.3. *Projected traction-based failure criterion*

Regarding a given stress-based failure criterion $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$, the discontinuity angle θ^{cr} is determined from the kinematic constraints (4.35) for the dissipative flow tensor $\boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}} := \partial \widehat{F}/\partial \boldsymbol{\sigma}$. As $\widehat{\Lambda}_{pp} = 0$ and $\widehat{\Lambda}_{mp} = 0$ are automatically (always) satisfied in the plane stress condition, it follows from the remaining constraint $\widehat{\Lambda}_{pp}(\theta^{\mathrm{cr}}) = 0$ that

$$\cos(2\theta^{\mathrm{cr}}) = \frac{\widehat{\Lambda}_1 + \widehat{\Lambda}_2}{\widehat{\Lambda}_1 - \widehat{\Lambda}_2}, \qquad \sin(2\theta^{\mathrm{cr}}) = 2\mathrm{sign}(\sigma_{nm})\frac{\sqrt{-\widehat{\Lambda}_1\widehat{\Lambda}_2}}{\widehat{\Lambda}_1 - \widehat{\Lambda}_2} \tag{6.9}$$

if the conditions $\widehat{\Lambda}_1 \geq 0$ and $\widehat{\Lambda}_2 \leq 0$ are satisfied.

With the discontinuity angle θ^{cr} defined from Eq. (6.9), it follows from Eq. (4.33a) that

$$\gamma_n = \widehat{\Lambda}_{nn}(\theta^{\text{cr}}) = \big(\widehat{\Lambda}_1 - \widehat{\Lambda}_2\big)\cos(2\theta^{\text{cr}}) = \widehat{\Lambda}_1 + \widehat{\Lambda}_2 \tag{6.10a}$$

$$\gamma_m = 2\widehat{\Lambda}_{nm}(\theta^{\text{cr}}) = \big(\widehat{\Lambda}_1 - \widehat{\Lambda}_2\big)\sin(2\theta^{\text{cr}}) = 2\text{sign}(\sigma_{nm})\sqrt{-\widehat{\Lambda}_1\widehat{\Lambda}_2} \tag{6.10b}$$

The traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$ projected from the stress-based counterpart $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$ is then obtained from Eq. (4.55) as

$$f(\boldsymbol{t}, q) = \frac{1}{M}\Big[(\widehat{\Lambda}_1 + \widehat{\Lambda}_2)t_n + 2\sqrt{-\widehat{\Lambda}_1\widehat{\Lambda}_2}\,\big|t_m\big| - \widehat{h}\cdot q\Big] \leq 0 \tag{6.11}$$

for the normal and tangential tractions (t_n, t_m).

Remark 6.1 If the discontinuity angle θ^{cr} determined from Eq. (6.9) exists, it can be verified that, the following stationarity conditions associated with the projected traction-based failure criterion (6.11) are satisfied

$$\frac{\partial f}{\partial \theta}\bigg|_{\theta^{\text{cr}}} = 0, \qquad \frac{\partial^2 f}{\partial \theta^2}\bigg|_{\theta^{\text{cr}}} < 0 \tag{6.12}$$

Namely, the tractions (t_n, t_m) on the orientation $\boldsymbol{n}(\theta^{\text{cr}})$ maximize the traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$. □

Remark 6.2 For the case $\widehat{\Lambda}_2 > 0$ or $\widehat{\Lambda}_1 < 0$, Eq. (6.9) does not hold and the projection relation (4.53) cannot be fulfilled for the given stress-based failure criterion $\widehat{f}(\boldsymbol{\sigma}, q) \leq 0$. As mentioned in Remark 5.1, some modifications should be made in this situation so that the discontinuity angle is determined as in the traction-based models, i.e., $\theta^{\text{cr}} = 0$ if $\widehat{\Lambda}_1 > \widehat{\Lambda}_2 > 0$ and $\theta^{\text{cr}} = \pi/2$ if $\widehat{\Lambda}_2 < \widehat{\Lambda}_1 < 0$; see Wu and Cervera [66] for the details. □

Remark 6.3 For a softening material with associated evolution laws, in the plane stress condition the discontinuity angle determined from Eq. (6.9) coincides with that obtained from the classical discontinuous bifurcation analysis [55]. However, for the cases of non-associated evolution laws or plane strain they are in general different. □

6.4. *A generic failure criterion*

In this section a generic stress-based failure criterion is considered in the plane stress condition. Depending on the involved material parameters, the failure surface defines either an ellipse, a parabola, a hyperbola or the product of two straight lines. The classical von Mises, Drucker-Prager and Mohr-Coulomb criteria are also recovered as particular examples. In all cases, the discontinuity angles and the associated traction-based failure criteria are derived explicitly.

6.4.1. *Stress-based failure criterion*

Without loss of generality, let us consider the following stress-based failure criterion $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$

$$\widehat{F}(\boldsymbol{\sigma}, q) = J_2 - \frac{1}{6}A_0 I_1^2 + \frac{1}{3}B_0 q I_1 - C_0 q^2 \leq 0 \tag{6.13}$$

where the non-negative parameters B_0 and C_0 are related to the parameter $A_0 < 2$ through

$$B_0 = \frac{2 - A_0}{2}\big(\rho - 1\big) \geq 0, \qquad C_0 = \frac{2 - A_0}{6}\rho \geq 0 \tag{6.14}$$

with the ratio $\rho := f_c/f_t \geq 1$ between the uniaxial compressive and tensile strengths f_c and f_t. The residual strength $q(\kappa)$ is normalized so that its initial value is $q^0 = f_t$. It can be verified that the above failure criterion $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$ is a homogeneous function of degree $M = 2$.

In the plane stress condition ($\sigma_3 = 0$), the failure criterion (6.13) becomes

$$\widehat{F}(\boldsymbol{\sigma}, q) = \frac{2 - A_0}{6}(\sigma_1^2 + \sigma_2^2) - \frac{1 + A_0}{3}\sigma_1\sigma_2 + \frac{B_0}{3}(\sigma_1 + \sigma_2)q - C_0 q^2 \leq 0 \tag{6.15}$$

in the $\sigma_1 - \sigma_2$ principal stress space. For the model parameter $A_0 < 2$ the failure surface $\widehat{F}(\boldsymbol{\sigma}, q) = 0$ is less open than a right angle, which is the case interested here. Furthermore, only for the parameters satisfying $A_0 \leq 1/2 + B_0^2/(6C_0)$ can the failure surface $\widehat{F}(\boldsymbol{\sigma}, q) = 0$ intersect the hydrostatic axis $\sigma_1 = \sigma_2$. It then follows from the relations (6.14) that the admissible parameter A_0 is within the range

$$A_0 \leq \frac{2(\rho^2 - \rho + 1)}{(\rho + 1)^2} < 2 \tag{6.16}$$

Accordingly, the following function types can be defined for the failure criterion (6.15)

$$\begin{cases} A_0 < \frac{1}{2} & \text{Elliptical function; see Figure 9(a)} \\ A_0 = \frac{1}{2} & \text{Parabolic function; see Figure 10(a)} \\ \frac{1}{2} < A_0 < 2(\rho^2 - \rho + 1)/(\rho + 1)^2 & \text{Hyperbolic function; see Figure 11(a)} \\ A_0 = 2(\rho^2 - \rho + 1)/(\rho + 1)^2 & \text{Product of two straight lines; see Figure 12(a)} \end{cases} \tag{6.17}$$

Remark 6.4 For the failure criterion $\widehat{F}(\boldsymbol{\sigma}, q) \leq 0$ in Eq. (6.13), the associated dissipative flow tensor $\widehat{\boldsymbol{\Lambda}} := \partial \widehat{F}/\partial \boldsymbol{\sigma}$ and its principal values $\widehat{\Lambda}_i := \partial \widehat{F}/\partial \sigma_i$ $(i = 1, 2, 3)$ are expressed as

$$\widehat{\boldsymbol{\Lambda}} = \boldsymbol{s} + \frac{1}{3}\big(B_0 q - A_0 I_1\big)\boldsymbol{I}, \qquad \widehat{\Lambda}_i = s_i + \frac{1}{3}\big(B_0 q - A_0 I_1\big) \tag{6.18}$$

together with the function $\widehat{h} := -\partial \widehat{F}/\partial q = -\frac{1}{3}B_0 I_1 + 2C_0 q$, where σ_i and s_i denote the principal values of the stress tensor $\boldsymbol{\sigma}$ and its deviatoric part $\boldsymbol{s}$, respectively. Moreover, it follows from Eqs. (6.18) that

$$\widehat{\Lambda}_1 - \widehat{\Lambda}_2 = s_1 - s_2 = \sigma_1 - \sigma_2 \tag{6.19}$$

Accordingly, for the discontinuity angle θ^{cr} determined from Eq. (6.9), the tangential traction t_m and dissipative flow component γ_m are evaluated as

$$t_m = \sigma_{nm} = \text{sign}(t_m)\sqrt{-\widehat{\Lambda}_1\widehat{\Lambda}_2}, \qquad \gamma_m = 2t_m \tag{6.20}$$

The above results are useful in determination of the projected traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$. □

Remark 6.5 The classical Drucker-Prager criterion is recovered for the model parameters satisfying $A_0 = B_0^2/(6C_0)$, or, equivalently,

$$A_0 = 2\alpha^2, \qquad B_0 = 2\alpha\big(1 + \alpha\big), \qquad C_0 = \frac{1}{3}\big(1 + \alpha\big)^2 \tag{6.21}$$

for the material constant $\alpha := (\rho - 1)/(\rho + 1) \in [0, 1)$ dependent on the strength ratio ρ. In particular, von Mises criterion corresponds to $\alpha = 0$. □

Remark 6.6 In the plane stress condition the model parameters satisfying $B_0^2 = 3(2A_0 - 1)C_0$ are determined as

$$A_0 = \frac{2(\rho^2 - \rho + 1)}{(\rho + 1)^2}, \qquad B_0 = \frac{3\rho(\rho - 1)}{(\rho + 1)^2}, \qquad C_0 = \frac{\rho^2}{(\rho + 1)^2} \tag{6.22}$$

Accordingly, the failure criterion (6.13) becomes

$$\hat{F}(\boldsymbol{\sigma}, q) = \frac{\rho\sigma_1 - \sigma_2 - \rho q}{\rho + 1} \cdot \frac{\sigma_1 - \rho\sigma_2 + \rho q}{\rho + 1} \leq 0 \tag{6.23}$$

with the left branch interested coincides with the classical Mohr-Coulomb criterion, i.e.,

$$\frac{\rho\sigma_1 - \sigma_2 - \rho q}{\rho + 1} = \frac{1}{2}\Big[\big(\sigma_1 + \sigma_2\big)\sin\varphi + \big(\sigma_1 - \sigma_2\big) - q\big(1 + \sin\varphi\big)\Big] \leq 0 \tag{6.24}$$

where the internal friction angle $\varphi \in [0, \pi/2]$ of the material is given by

$$\sin\varphi = \frac{\rho - 1}{\rho + 1} \qquad \Longleftrightarrow \qquad \rho = \frac{1 + \sin\varphi}{1 - \sin\varphi} \tag{6.25}$$

Note that the Mohr-Coulomb criterion (6.24) applies for $\sigma_1 \geq 0$ and $\sigma_2 \leq 0$ in the plane stress state ($\sigma_3 = 0$). □

6.4.2. Discontinuity angle and traction-based failure criterion

For a discontinuity (band) to form, the failure criterion (6.15) is activated. That is, the residual strength $q \in [0, f_t]$ can be expressed in terms of the principal stress σ_1

$$\hat{F}(\boldsymbol{\sigma}, q) = 0 \qquad \Longrightarrow \qquad q = \alpha_r \sigma_1 \tag{6.26}$$

where the coefficient $\alpha_r(r) \in [0, 1]$ depends on the stress ratio $r := \sigma_2/\sigma_1$

$$\alpha_r = \frac{B_0\big(1 + r\big) \pm \mathrm{sign}(\sigma_1)\sqrt{B_0^2\big(1 + r\big)^2 + 6A_r C_0}}{6C_0} \tag{6.27a}$$

$$A_r = \big(2 - A_0\big) - 2\big(1 + A_0\big)r + \big(2 - A_0\big)r^2 \tag{6.27b}$$

The discontinuity angle θ^{cr} is then given from Eq. (6.9) by

$$\cos(2\theta^{\mathrm{cr}}) = \frac{2B_0 q + \big(1 - 2A_0\big)\big(\sigma_1 + \sigma_2\big)}{3\big(\sigma_1 - \sigma_2\big)} = \frac{2B_0\alpha_r + \big(1 - 2A_0\big)\big(1 + r\big)}{3\big(1 - r\big)} \tag{6.28}$$

Clearly, it depends on the stress ratio $r := \sigma_2/\sigma_1$ which remains fixed for a proportional load path.

For the discontinuity angle (6.28), it follows from Eqs. (6.1), (6.10) and $(6.18)_2$ that

$$\gamma_n = \frac{B_0 q - \big(2A_0 - 1\big)t_n}{2 - A_0}, \qquad \hat{h} = -\frac{B_0}{2 - A_0}t_n + \left[\frac{B_0^2}{3\big(2 - A_0\big)} + 2C_0\right]q \tag{6.29}$$

Recalling the results (6.20), the projected traction-based failure criterion $f(\boldsymbol{t}, q) \leq 0$, also homogeneous of degree $M = 2$, is determined from Eq. (6.11) as

$$f(\boldsymbol{t}, q) = t_m^2 - \frac{1}{2 - A_0}\left[\Big(A_0 - \frac{1}{2}\Big)t_n^2 - B_0 q t_n + \frac{1}{6}B_0^2 q^2\right] - C_0 q^2 \leq 0 \tag{6.30}$$

For the admissible parameter $A_0 \leq 2(\rho^2 - \rho + 1)/(\rho + 1)^2$, the following cases can be identified:

- $A_0 < \frac{1}{2}$: As shown in Figure 9(b), the failure criterion (6.30) defines an ellipse on the $t_n - t_m$ plane

$$f(\boldsymbol{t}, q) = t_m^2 + \frac{\frac{1}{2} - A_0}{2 - A_0}\left(t_n + \frac{B_0}{1 - 2A_0} q\right)^2 - \left[\frac{B_0^2}{3(1 - 2A_0)} + C_0\right] q^2 \leq 0 \tag{6.31}$$

with its foci on the axis t_n and centered at $t_n = -B_0 q / (1 - 2A_0)$. An interesting particular case corresponds to $\rho = 1.0$, i.e., $B_0 = 0$ and $C_0 = (2 - A_0)/6$. In this case, the failure criterion (6.31) becomes

$$f(\boldsymbol{t}, q) = t_m^2 + \frac{\frac{1}{2} - A_0}{2 - A_0} t_n^2 - \frac{2 - A_0}{6} q^2 \leq 0 \tag{6.32a}$$

or, equivalently,

$$\sqrt{t_n^2 + \beta^{-2}\, t_m^2} - \frac{2 - A_0}{\sqrt{3(1 - 2A_0)}} q \leq 0 \tag{6.32b}$$

for the parameter $\beta := \sqrt{(\frac{1}{2} - A_0)/(2 - A_0)} \in [0, 1]$. The failure criterion (6.32b) is exactly the one suggested by Camacho and Ortiz [6], Pandolfi et al. [47], Jirásek and Zimmermann [24] for modeling mixed-mode failure.

- $A_0 = \frac{1}{2}$: As shown in Figure 10(b), the failure criterion (6.30) is a parabola on the $t_n - t_m$ plane

$$f(\boldsymbol{t}, q) = t_m^2 + \beta_1 q t_n - \beta_2 q^2 \leq 0 \tag{6.33}$$

where the parameters β_1 and β_2 are expressed as

$$\beta_1 = \frac{2}{3} B_0 = \frac{1}{2}(\rho - 1), \qquad \beta_2 = \frac{1}{9} B_0^2 + C_0 = \frac{1}{16}(\rho + 1)^2 \tag{6.34}$$

The parabolic traction-based failure criterion has been adopted in the embedded discontinuity models [32, 35].

- $\frac{1}{2} < A_0 < 2(\rho^2 - \rho + 1)/(\rho + 1)^2$, or equivalently, $B_0^2 > 3(2A_0 - 1)C_0$: As shown in Figure 11(b), the failure criterion (6.30) defines a hyperbola on the $t_n - t_m$ plane, with the left branch of interest expressed as

$$\tan\varphi \cdot t_n + \sqrt{t_m^2 + \omega^2 q^2} - c \leq 0 \tag{6.35}$$

where $\tan\varphi$ and c denote the friction coefficient and cohesion of the asymptotic Mohr-Coulomb criterion

$$\tan\varphi = \sqrt{\frac{A_0 - 1/2}{2 - A_0}}, \qquad c = \frac{B_0 q}{\sqrt{2(2 - A_0)(2A_0 - 1)}} \tag{6.36a}$$

together with the following parameter ω

$$\omega = \sqrt{\frac{B_0^2}{3(2A_0 - 1)} - C_0} \tag{6.36b}$$

Similar hyperbolic failure function has been adopted in the literature to describe the normal and shear coupling of cohesive cracks [7, 14, 33, 59, 62]; see Section 6.4.3 for further discussion.

- $A_0 = 2(\rho^2 - \rho + 1)/(\rho + 1)^2$, or equivalently, $B_0^2 = 3(2A_0 - 1)C_0$: As shown in Figure 12(b), the failure function (6.30) becomes the product of two straight lines on the $t_n - t_m$ plane, with the left branch of interest corresponding to $\omega = 0$ in the hyperbolic criterion (6.35)

  $$\tan\varphi \cdot t_n + |t_m| - c \le 0 \tag{6.37}$$

  where the internal friction coefficient $\tan\varphi$ and the cohesion c are given by

  $$\tan\varphi = \frac{1}{2}\,\frac{\rho - 1}{\sqrt{\rho}}, \qquad c = \frac{1}{2}\sqrt{\rho}\, q \tag{6.38}$$

  This is exactly the classical Mohr-Coulomb criterion.

As can be seen, the function types of the projected traction-based failure criterion $f(\boldsymbol{t}, q) \le 0$ coincide with those of the stress-based failure criterion $\widehat{F}(\boldsymbol{\sigma}, q) \le 0$ classified in Eq. (6.17).

Remark 6.7 The traction-based failure criterion projected from the classical Drucker-Prager model can be obtained by substituting the model parameters (6.21) into Eq. (6.30), i.e.,

$$f(\boldsymbol{t}, q) = t_m^2 - \frac{4\alpha^2 - 1}{4\big(1 - \alpha^2\big)} t_n^2 + \frac{\alpha}{1 - \alpha} q t_n - \frac{1 + \alpha}{3\big(1 - \alpha\big)} q^2 \le 0 \tag{6.39}$$

Depending on the value of the parameter $\alpha := \sqrt{A_0/2} \in [0, 1)$, the traction-based failure criterion (6.39) defines: (*i*) an ellipse for $0 \le \alpha < 1/2$ or, equivalently, $0 \le A_0 < 1/2$; (*ii*) a parabola for $\alpha = 1/2$ or, equivalently, $A_0 = 1/2$, and (*iii*) a hyperbola for $1/2 < \alpha < 1$ or, equivalently, $1/2 < A_0 < 2(\rho^2 - \rho + 1)/(\rho + 1)^2$, respectively, on the $t_n - t_m$ plane. Note that the condition $A_0 = 2(\rho^2 - \rho + 1)/(\rho + 1)^2$, corresponding to $B_0^2 = 3(2A_0 - 1)C_0$, cannot be reached for Drucker-Prager criterion in the case of plane stress. □

6.4.3. Further discussion on the hyperbolic failure criterion

In the literature [7, 14, 33, 59, 62], different forms of hyperbolic traction-based failure criteria have been proposed for the modeling of localized failure in quasi-brittle materials, owing to the excellent data fitting capabilities and other advantageous features (e.g., no slope discontinuity at the tip, asymptotically approaching to Mohr-Coulomb criterion for increasing compression, and a constant friction angle, etc.). However, the involved parameters are essentially mesoscopical entities hard to be determined from standard experimental tests. This shortcoming restrains heavily its application. In previous sections we have established the theoretical connections and equivalence conditions between localized models with traction- and stress-based failure criteria. Such a correspondence facilitates identifying those mesoscopic parameters involved in the traction-based failure criterion from the easily obtained macroscopic ones.

Without loss of generality, the following traction-based failure criterion [7, 62] is considered

$$\hat{f}(\boldsymbol{t}, q) = t_m^2 - \big(c - t_n \cdot \tan\varphi\big)^2 + \big(c - \chi \cdot \tan\varphi\big)^2 \le 0 \tag{6.40}$$

with the left branch of interest expressed as [33]

$$\tan\varphi \cdot t_n + \sqrt{t_m^2 + \big(c - \chi \cdot \tan\varphi\big)^2} - c \le 0 \tag{6.41}$$

where χ denotes the failure strength, not necessarily identical to the macroscopic material tensile strength f_t; $\tan\varphi$ and c represent the friction and cohesion of the asymptotic Mohr-Coulomb criterion. All these parameters cannot be determine easily and in general are assumed arbitrarily [7, 14, 62].

As shown in Section 6.4.2, for the model parameter $\frac{1}{2} < A_0 < 2(\rho^2 - \rho + 1)/(\rho + 1)^2$, in the condition of plane stress an inelastic softening solid characterized by the stress-based failure criterion (6.13) localizes into a discontinuity (band) described by the traction-based failure criterion (6.35) of identical form to Eq. (6.41). Obviously, the equivalence between them implies that

$$\omega q = c - \chi \cdot \tan\varphi \qquad \Longleftrightarrow \qquad \chi = \frac{c - \omega q}{\tan\varphi} \tag{6.42}$$

where the friction coefficient $\tan\varphi$, the cohesion c and the parameter ω are given in Eq. (6.36).

Reversely, recalling the tractions (6.1) and making some straightforward manipulations, the stationarity condition (6.3) for the failure criterion (6.40) yields the following discontinuity angle $\widehat{\theta}^{\text{cr}}$

$$\cos(2\widehat{\theta}^{\text{cr}}) = \frac{\tan\varphi}{1 + \tan^2\varphi} \cdot \frac{2c - (\sigma_1 + \sigma_2)\tan\varphi}{\sigma_1 - \sigma_2} \tag{6.43}$$

In accordance with Eqs. (6.1), the normal and tangential tractions (t_n, t_m) acting on the orientation $\boldsymbol{n}(\widehat{\theta}^{\text{cr}})$ can be expressed in terms of the principal stresses σ_1 and σ_2. Substitution of the obtained results into Eq. (6.40) yields

$$F(\boldsymbol{\sigma}, q) = \frac{(\sigma_1 - \sigma_2)^2}{4} - \frac{1}{1 + \tan^2\varphi}\Big(c - \frac{\sigma_1 + \sigma_2}{2}\tan\varphi\Big)^2 + \big(c - \chi \cdot \tan\varphi\big)^2 \le 0 \tag{6.44}$$

With the relations (6.36) and (6.42), the discontinuity angle (6.28) are recovered and the reconstructed stress-based failure criterion (6.44) coincides with the given counterpart (6.15).

Therefore, a localized model with the stress-based failure criterion (6.15) is indeed equivalent to that with the given traction-based failure criterion (6.40), providing the involved parameters are consistently connected as in Eqs. (6.36) and (6.42).

As an example, let us consider a strength ratio $\rho := f_c/f_t = 10$, typical for concrete. In order for a discontinuity (band) characterized by the hyperbolic traction-based failure criterion (6.35) to form in the plane stress condition, the model parameter A_0 has to be in the range $A_0 \in (0.5, 1.504)$. As shown in Figure 13(a), the biaxial strength envelope corresponding to the following parameters

$$A_0 = 1.48 \qquad \Longrightarrow \qquad B_0 = 2.34, \qquad C_0 = 0.93 \tag{6.45}$$

fits the test data of normal concrete in tension-tension and tension-compression rather well. It immediately follows from the relations (6.36) and (6.42) that

$$\tan\varphi = 1.37, \qquad c = 1.64q, \qquad \chi = q \tag{6.46}$$

The corresponding traction-based failure criterion is shown in Figure 13(b). Furnished with an appropriate softening law $q(\kappa)$, it can be used in the modeling of localized failure in concrete. This topic will be reported elsewhere.

7. Conclusions

This paper addresses traction- and stress-based approaches for the modeling of strong/regularized discontinuities induced localized failure in solids. Different approaches resulting from two complementary methodologies, i.e., discontinuities localized in an elastic solid and strain localization of an inelastic solid are systematically investigated.

In the first methodology, discontinuities are localized in elastic solids along a known orientation. It is assumed *a priori* that the bulk material remains elastic during the whole deformation process and all inelastic behavior is localized within the discontinuity (band). That is, strain localization always occurs with a continuous stress field. A traction-based failure criterion can be introduced to characterize the discontinuity (band). However, the discontinuity orientation cannot be determined uniquely unless some additional assumptions, e.g., Mohr's maximization postulate considered in this work, are made. With respect to the strategies dealing with the traction continuity condition between the bulk stresses and tractions across the discontinuity (band), two approaches can be identified. If the traction continuity condition is enforced in weak (average) form, the strong/regularized discontinuity approaches follow. In this approach, the displacement jumps (or equivalently, the inelastic deformations) are retained as independent variables, and constitutive models for both the bulk and discontinuity (band) should be fed. Alternatively, being enforced in strong (point-wise) form, the traction continuity condition coincides with the classical static constraint. This property allows eliminating the kinematic unknowns (i.e., displacement jumps or inelastic deformations) associated with the discontinuity (band) from the governing equations. In the resulting embedded/smeared discontinuity approaches, an overall inelastic constitutive model fulfilling the static constraint suffices.

In order to check whether above strain localization can occur and identify its consequences on the resulting approaches, the second methodology is considered. The kinematic constraint guaranteeing stress boundedness/continuity upon strain localization is established for general inelastic softening solids. It is then applied to a unified elastoplastic damage model with the inelastic evolution laws characterized by a dissipative flow tensor. For the discontinuity kinematics to be reproduced in an appropriate manner, the tensorial components of the dissipative flow tensor in the directions orthogonal to the discontinuity orientation have to vanish upon strain localization. Satisfaction of this kinematic constraint allows introducing a traction-based failure criterion and developing a localized cohesive model for the discontinuity (band), justifying the first methodology. Dependent on the strategies dealing with the discontinuity orientation and failure criterion, two dual (formally identical) but not necessarily equivalent approaches are identified. For a given traction-based failure criterion and a known discontinuity orientation, the resulting elastoplastic damage model is equivalent to the embedded/smeared discontinuity models discussed before. Alternatively, for a given stress-based failure criterion, the discontinuity orientation and associated traction-based counterpart are consistently determined from the kinematic constraint, resulting in the projected discontinuity model.

The *bi-directional* connections between the aforementioned traction- and stress-based approaches are explored in general 3D cases. Particularly, the equivalence between Mohr's maximization postulate for a traction-based failure criterion and the kinematic constraint for a stress-based one is established. This equivalence makes it possible to bridge the gap between traction- and stress-based approaches. In the plane stress condition, the discontinuity orientation and the corresponding stress-/traction-based failure criteria are explicitly given in closed form. A generic stress-based failure criterion is then analyzed in a unified manner. Depending on the involved parameters, the projected traction-based failure criterion can be either an ellipse, a parabola, a hyperbola or the product of two straight lines, recovering most of those adopted in the literature, e.g., the classical Von Mises (J_2), Drucker-Prager and Mohr-Coulomb criteria, as particular cases. In particular, a traction-based failure criterion of hyperbolic type, widely adopted for the modeling of mixed mode localized failure in quasi-brittler materials, is obtained naturally. More importantly, owing to the established equivalence, the involved model parameters can be identified from available standard experimental data, so that the deficiency hindering its application in practice is largely removed.

Only localized models with associated evolution laws are investigated in this work, though the stress boundedness/continuity condition and the resulting kinematic constraint also apply for non-associated cases. However, they are no longer equivalent to Mohr's maximization postulate, but to Roscoe's zero-extension postulate [45, 50]. Extension of the current work to localized models with non-associated evolution laws will be addressed in the future.

Acknowledgments

The support from the National Natural Science Foundation of China (51222811, 51008130) and the Fundamental Research Funds for the Central University (2014XZZX003-13) to the first author (J.Y. Wu) is acknowledged.

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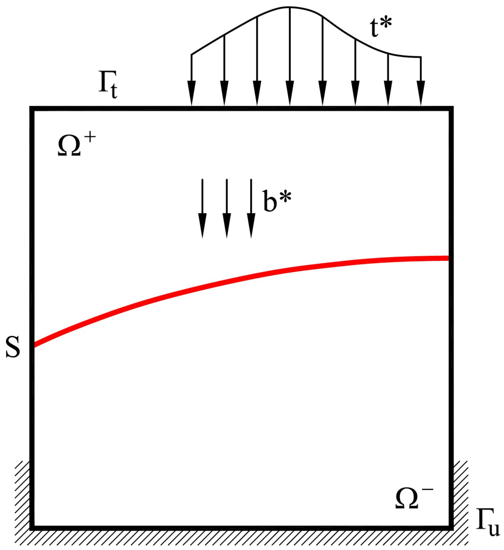
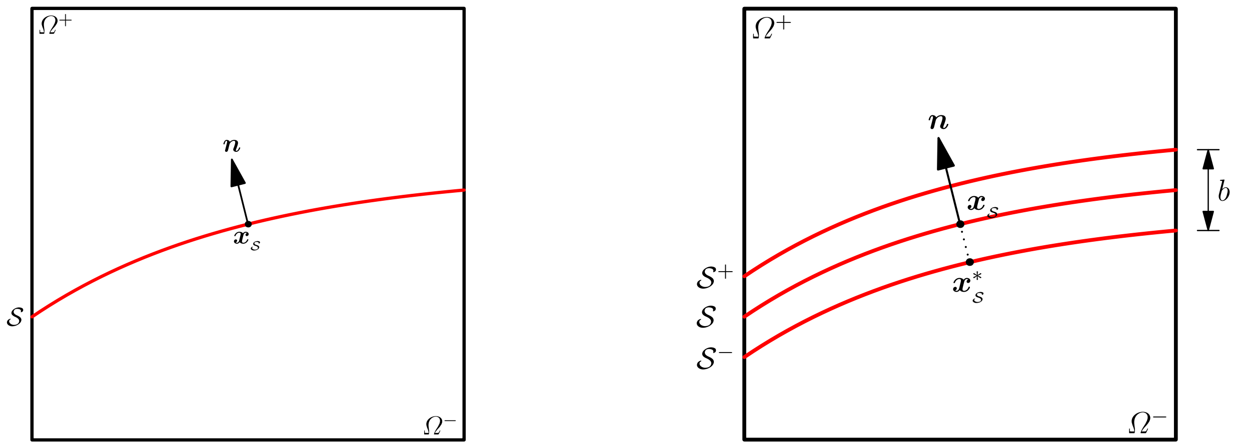


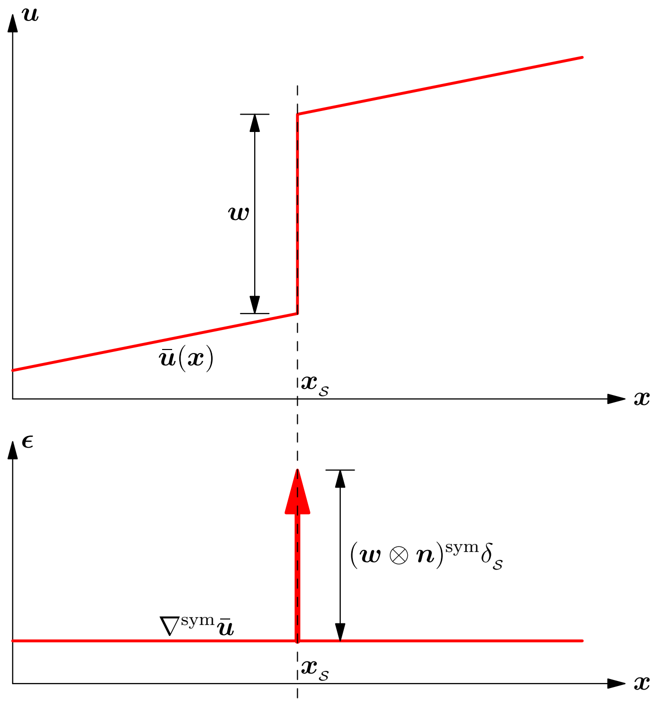
Figure 1: Problem setting in an elastic solid medium with an internal discontinuity



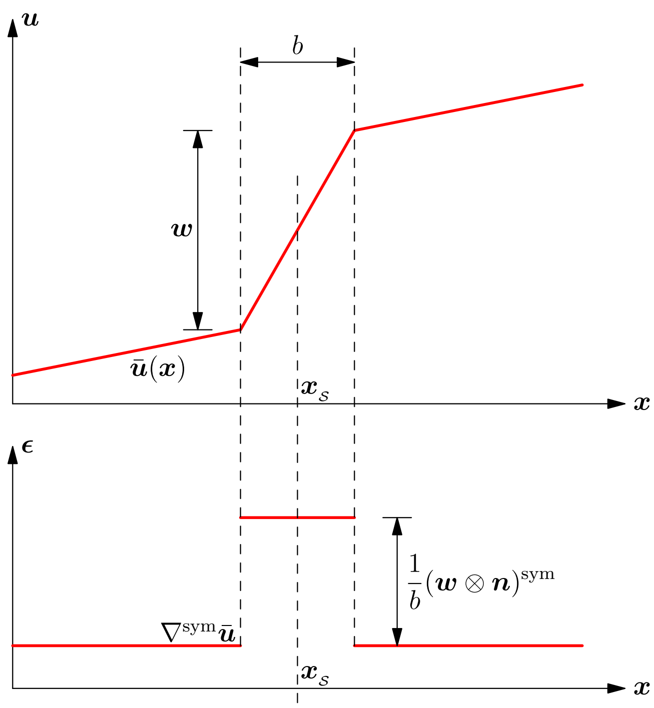
(a) Strong discontinuity

(b) Regularized discontinuity

Figure 2: Strong and regularized discontinuities in a solid



(a) Displacement/strain fields around a strong discontinuity



(b) Displacement/strain fields around a regularized discontinuity

Figure 3: Kinematics of strong/regularized discontinuities

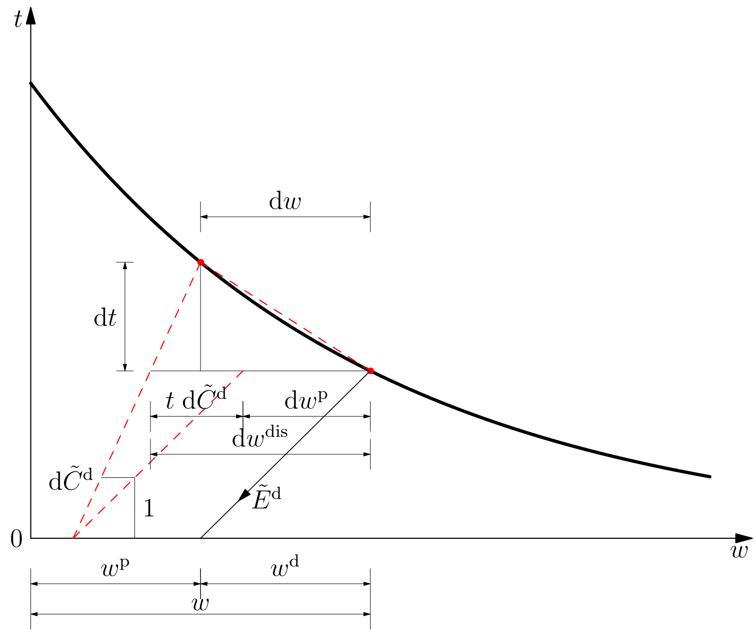


Figure 4: 1D definition of the dissipative jump rate and its damage/plastic components for an increment deformation

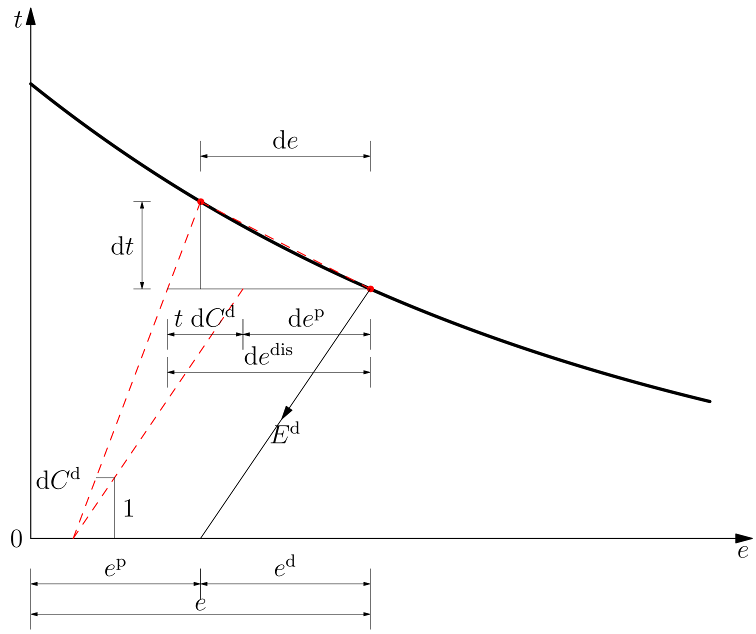


Figure 5: 1D definition of the dissipative deformation rate and its damage/plastic components for an increment deformation

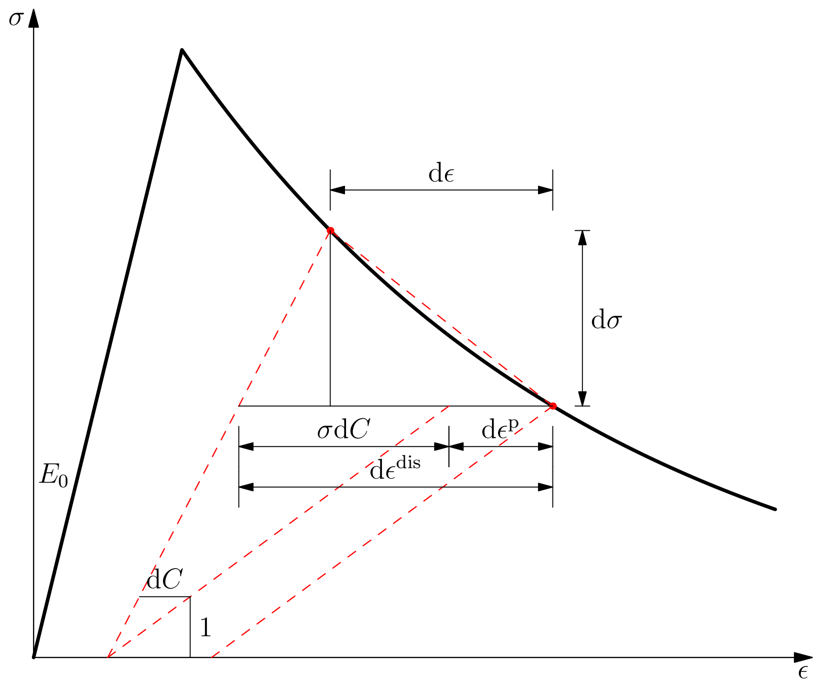


Figure 6: 1D definition of the dissipative strain rate and its damage/plastic components for an infinitesimal increment strain

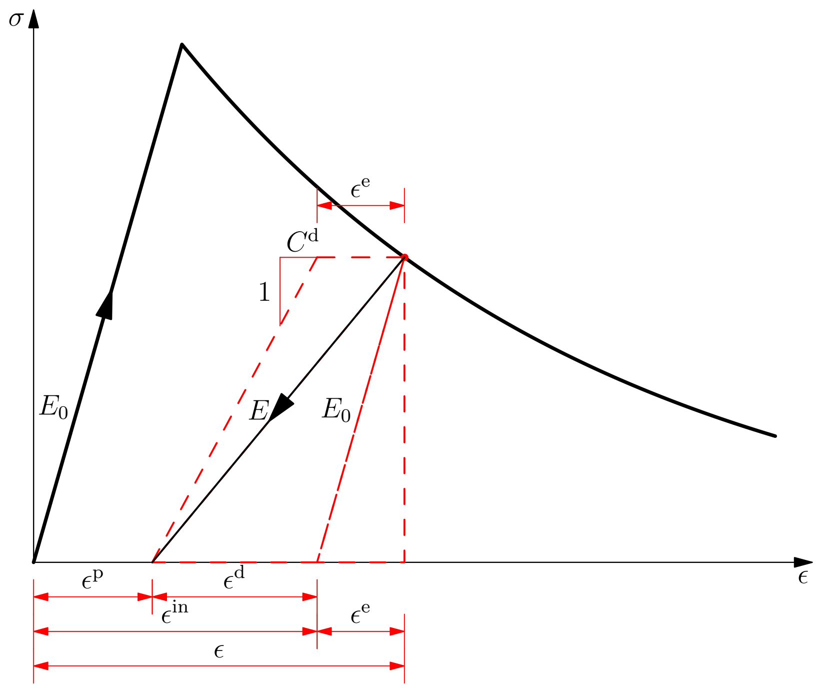


Figure 7: 1D stress vs. strain relation of the elastoplastic damage model: different kinematic decompositions

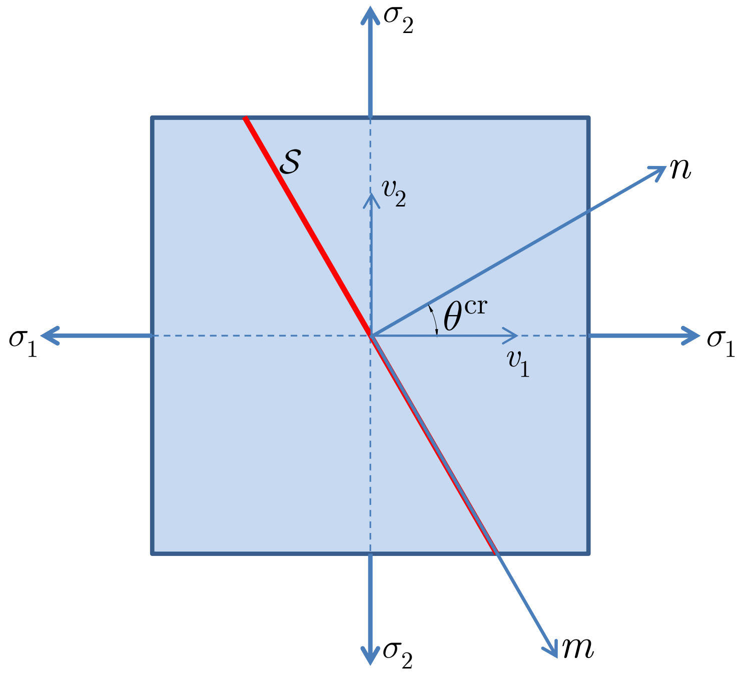
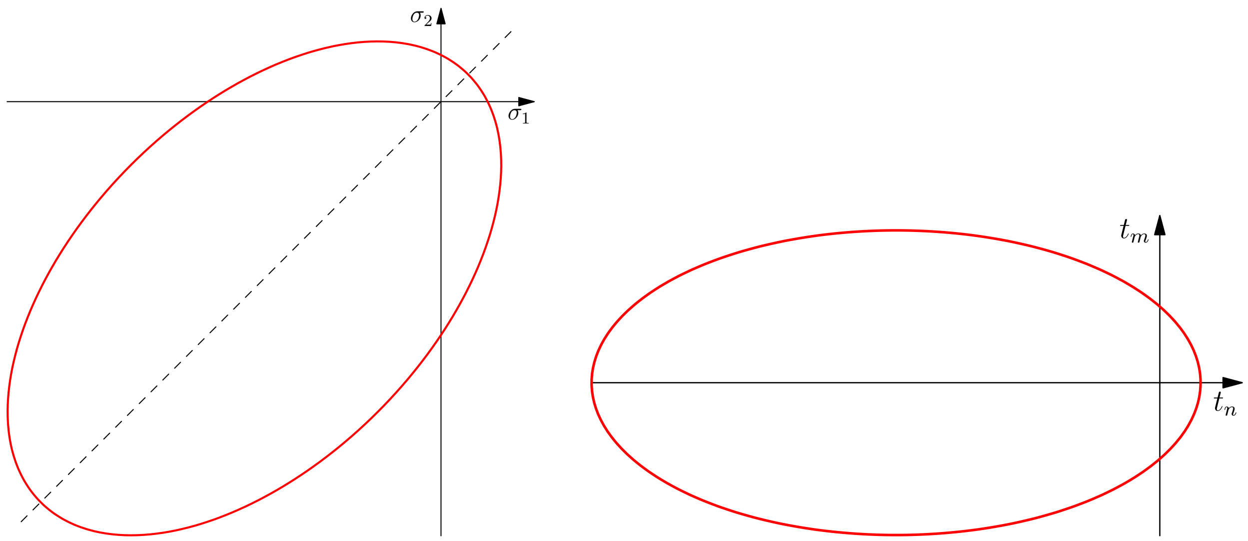


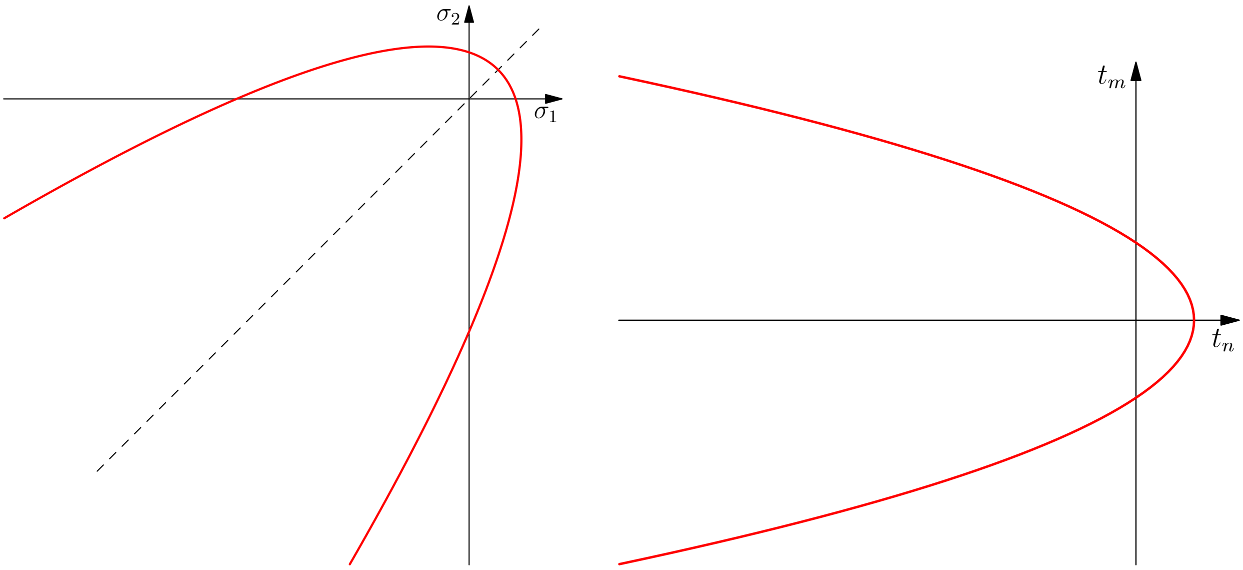
Figure 8: Definition of the discontinuity angle in 2D cases



(a) Stress-based failure surface

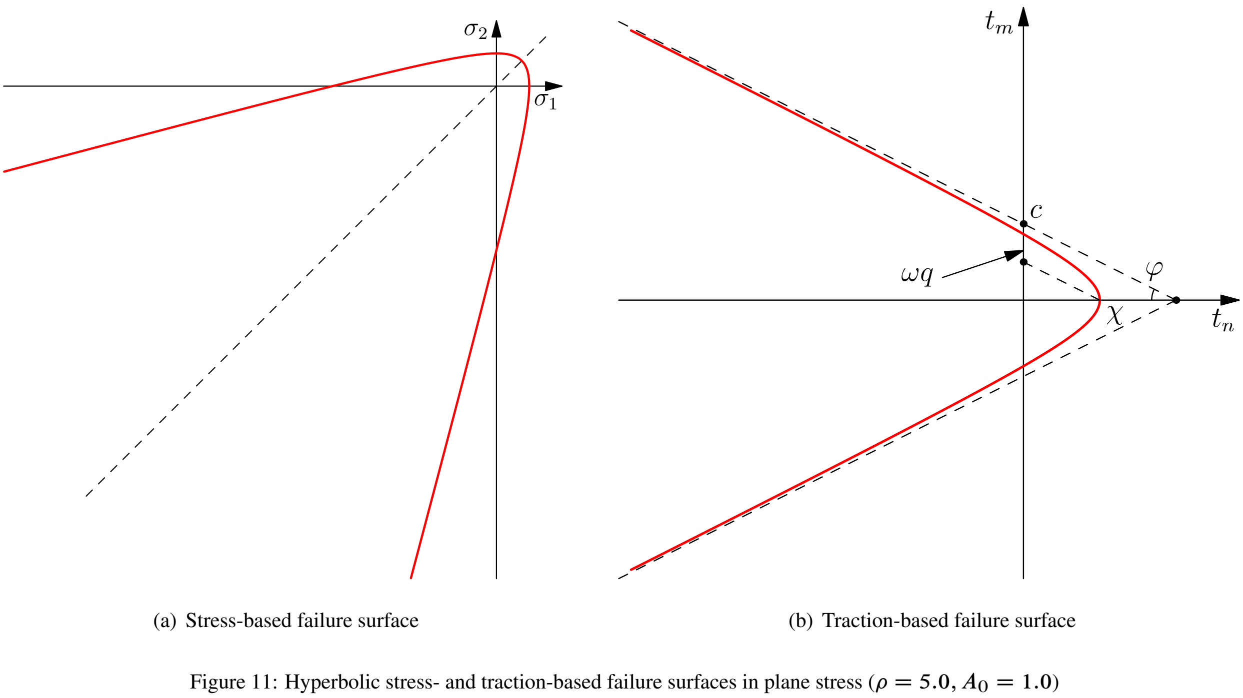
(b) Traction-based failure surface

Figure 9: Elliptic stress- and traction-based failure surfaces in plane stress ($\rho = 5.0$, $A_0 = 0.0$)



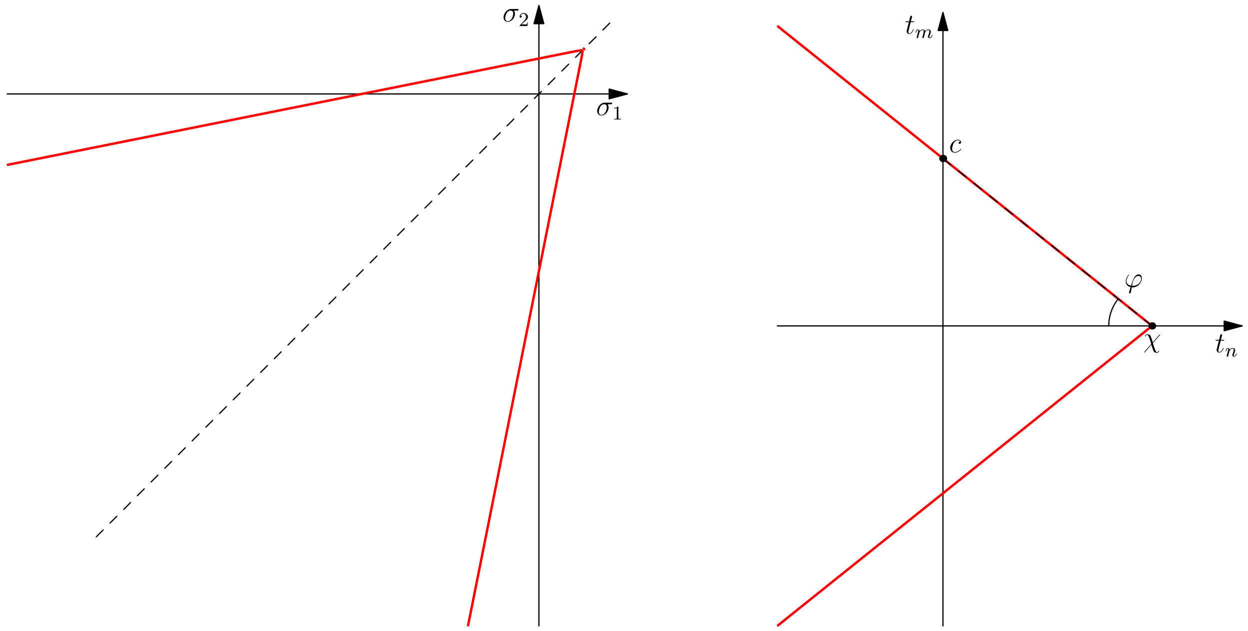
(a) Stress-based failure surface (b) Traction-based failure surface

Figure 10: Parabolic stress- and traction-based failure surfaces in plane stress ($\rho = 5.0$, $A_0 = 0.5$)



(a) Stress-based failure surface (b) Traction-based failure surface

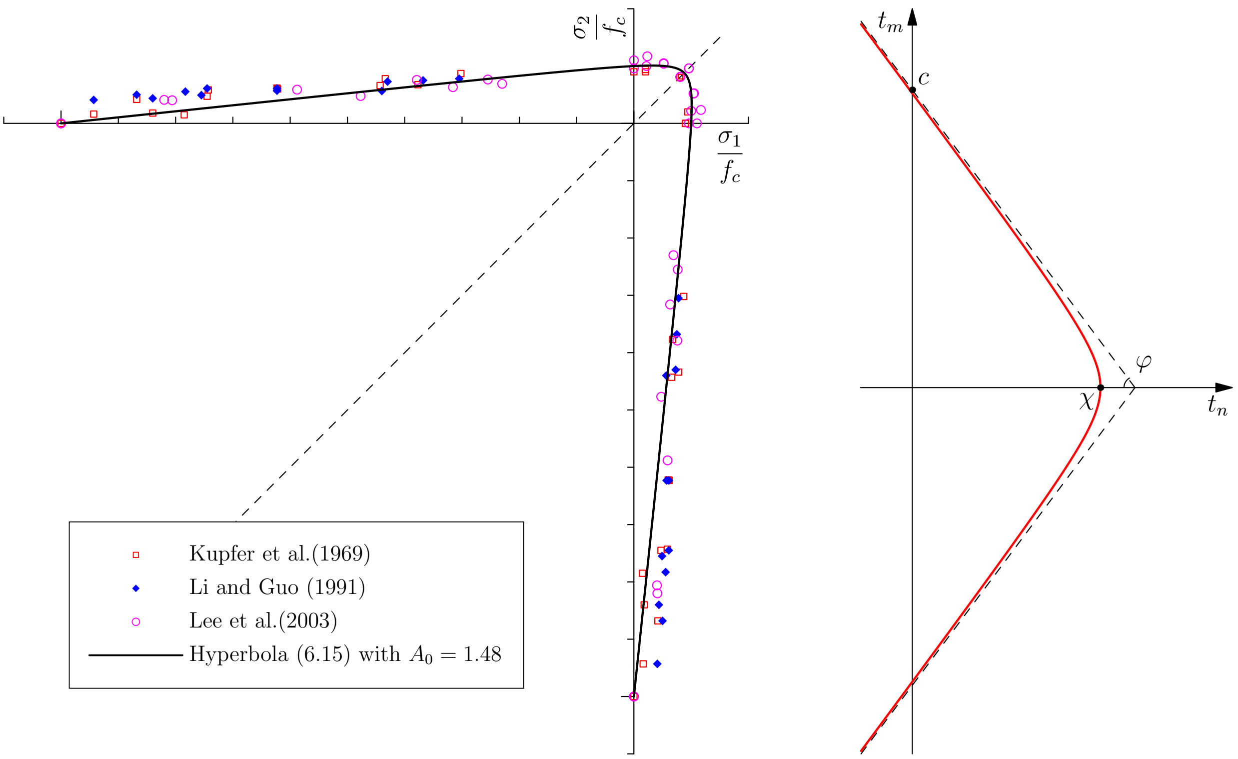
Figure 11: Hyperbolic stress- and traction-based failure surfaces in plane stress ($\rho = 5.0$, $A_0 = 1.0$)



(a) Stress-based failure surface

(b) Traction-based failure surface

Figure 12: Bilinear stress- and traction-based failure surfaces in plane stress ($\rho = 5.0, A_0 = 7/6$)



(a) Biaxial strength envelope

(b) Hyperbolic traction-based failure surface

Figure 13: Stress- and traction-based failure surface typical for concrete in plane stress